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Comparative Statics for Difference-in-Differences

Finn Christensen*

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Abstract

In a model with spillovers, consider the difference in the impact of a positive shock on average outcomes among the treated and untreated, or the marginal average effect of treatment among the treated with spillovers (MATTS). MATTS is positive if, and only if, the negated Jacobian inverse of the equilibrium system is a *B*-matrix by columns. This condition is also sufficient to answer traditional comparative statics questions. I also give several sufficient, and sometimes necessary, conditions on the noninverted Jacobian–which correspond to common modeling assumptions–under which its inverse is a *B*-matrix by columns. Sign restrictions on MATTS are testable because the sample difference-in-differences is an unbiased estimator for MATTS from a superpopulation perspective. I also show that MATTS generalizes the ATT and the ATE when interference, or spillovers, are present. I apply the results to oligopoly and contests.

JEL Codes: C31, C33, C65, C72, D21, L21

Keywords: Comparative statics, difference-in-differences, interference, spillovers, profit maximization hypothesis, *B*-matrix, symmetric games

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1 Introduction

When does a treatment or shock have a larger average impact on the treated than the untreated? This paper addresses this question, which helps predict whether a tax on some firms will reduce their market share, whether a decrease in effort costs for some contestants will increase their average relative effort, and whether an intervention to help the poor will raise their average relative incomes, among other outcomes.

I show that this comparative static, called the marginal average effect of treatment among the treated with spillovers (MATTS) generalizes the average treatment effect among the treated (ATT) when interference is present. Also, MATTS is identified by the population difference-in-differences (DiD) and, if assignment to treatment is (as if) random, the population difference-in-means (DiM). From a superpopulation perspective, and in contrast to many estimators for various estimands in the literature, the sample analogues are unbiased estimators even when spillovers are unrestricted.¹ This is significant because spillovers can lead to unexpected model behavior, making it essential when testing a model to use estimators whose unbiasedness remains robust in the presence of pervasive spillovers. This is crucial because economics offers numerous models to explain human and market behavior, many of which incorporate spillovers. Researchers would benefit if reduced form methods like DiD can effectively test and select appropriate models.

To my knowledge, existing equilibrium comparative statics results do not predict the sign of MATTS.² Spillovers introduce significant complexity. Some papers simplify the problem by considering the n = 2 case only. Other papers allow for n > 2 but examine the effect of a treatment or shock applied to a single individual. In the monotone case in which spillover effects are all positive or all negative, predictions are typically made for the outcome of each unit . None of these approaches is suitable for testing within the DiD framework, which estimates average effects and requires large treated and untreated groups for statistical consistency. Moreover, the results in

¹Depending on the context, spillover effects may be called equilibrium effects, strategic effects, network effects, indirect effects, social interactions, peer effects, or something else.

²This literature is large. Generally speaking it can be classified into approaches which assume differentiability and those that do not. Some examples assuming differentiability include Dixit (1986), Nti (1997), Acemoglu and Jensen (2013), Christensen and Cornwell (2018), Christensen (2019), and Norris et al. (2023), among others. Examples of lattice-theoretic or monotone methods include Topkis (2011), Milgrom and Roberts (1990), Amir (2005), Vives (1990), and L. C. Corchón (1994), among others.

this paper hold for *any* treatment group, so any statistically significant and unbiased estimate of MATTS which contradicts its predicted sign rejects the theory.

A second important contribution to this literature is facilitated by the fact that, within a single framework, I provide comparative statics results under the traditional assumptions mentioned above. In contrast, most papers in the literature explore the implications of just one, or a combination of some, of these assumptions. By comparing the sufficient conditions under each assumption we can more clearly see how the sign and heterogeneity of spillovers affect predictions.

The general setting involves any model with equilibrium characterized by a system of equations, interpreting each equation index as a unit of observation (e.g., firms). The Jacobian of the system encodes spillover effects and is involved when applying the Implicit Function Theorem (IFT). The (i, j) and (j, i) off-diagonal terms of the Jacobian capture *direct* spillover effects between units i and j while the off-diagonal terms of the Jacobian's inverse capture *equilibrium* spillover effects.

The central insight is that desirable comparative statics results obtain when the negated Jacobian inverse is a *B*-matrix by columns. *B*-matrices are defined in Section 2. This condition is necessary and sufficient for MATTS to increase with positive treatment for any treated subset (Theorem 2). In Corollary 1, I show that the same condition ensures that the average effect of treatment on the treated with spillovers (ATTS) increases, the sum of outcomes increases, and a singly treated unit's outcome increases more than any other unit's outcome changes.

While the conditions on the Jacobian inverse can be useful, oftentimes important modeling assumptions amount to restrictions on direct spillovers. I therefore find conditions on the elements of the (noninverted) Jacobian under which its negated inverse is a B-matrix by columns. In three of the four cases I consider, the conditions restrict the negated Jacobian to be a B-matrix (by rows).

Theorem 3 assumes direct spillovers are anonymous-by-unit, meaning that for each unit i, a change in unit j's outcome has the same spillover impact on unit i's outcome as a change in unit k's outcome. As illustrated in Section 7, these type of spillovers arise in models where a unit's objective function depends on others' outcomes only through their sum, and at symmetric equilibria of symmetric games. In this case, the negated Jacobian inverse is a *B*-matrix by columns if, and only if, the negated Jacobian is a *B*-matrix. Under this condition on direct spillovers, MATTS is positive and Corollary 1 applies. While we cannot generally sign the average spillover effect on the untreated (ASU), it is positive (negative) if all off-diagonal terms are positive (negative). These results subsume and expand on the results for a parameter shocks' impact on equilibrium variables in Dixit (1986) under weaker conditions.

Theorem 4 assumes positive direct spillovers, as in games with strategic complements. Here, every unit's outcome increases whenever the negated Jacobian is an M-matrix, a type of stability condition. Slightly stronger conditions ensure that the negated Jacobian inverse is a B-matrix by columns so that MATTS is positive.

Theorem 5 assumes negative direct spillovers, as in games with strategic substitutes. Under Willoughby's (1977) conditions, the negated Jacobian inverse is a B-matrix by columns and an M-matrix. In addition to implying that MATTS is strictly positive and that Corollary 1 applies, these conditions imply that treated units' outcomes increase (strictly positive ATTS), and untreated units' outcomes weakly decrease (negative ASU).

Theorem 6 formalizes the notion that intuitive comparative statics obtain with small spillovers, regardless of sign. This setting allows both positive and negative direct spillover effects within a unit but requires stronger conditions compared to when spillovers are all positive.³

These results are illustrated and applied in Section 7. First, I refine Silberberg's (1978) claim that the profit maximization hypothesis can be tested by observing whether a firm decreases output in response to a unit tax. I provide a perverse example in which firms are profit maximizers and a unit tax on some firms lowers every firm's output, yet MATTS is positive. However, MATTS must be negative if the market is perfectly competitive or spillovers are well-behaved in an oligopoly setting. Thus, a researcher must consider the market structure when using DiD methods to test the profit maximization hypothesis using Silberberg's approach.

The perverse example also highlights the importance of interpreting the sample DiD and, if assignment is (as if) random, DiM as an estimate of MATTS rather than an estimate of ATT or the average treatment effect (ATE) when spillovers are present. The researcher would wrongly reject the profit maximizing hypothesis if these estimates were misinterpreted.

Also in Section 7, in the context of contests I show that (near) symmetry gives rise to anonymous-by-unit spillovers at symmetric equilibria. This allows for powerful

 $^{^{3}}$ The anonymous-by-unit case allows direct spillovers to positive or negative depending on the unit, but they must be all positive or negative *within* a unit.

sign predictions on MATTS which can be tested in reduced form. As in most of the paper, the relevant literature is discussed in more detail in the relevant section.

Finally, while diagonally dominant matrices are central to existing IFT-based comparative statics, they play a secondary role here. Instead, I focus on *B*-matrices which were introduced in Carnicer et al. (1999) and Peña (2001), and were first applied to economics in Christensen (2019).⁴ Several new results for this class are provided in this paper, especially those which give conditions under which the inverse of a matrix is a *B*-matrix by columns.

As for a roadmap, the paper can be thought of as consisting of two complementary parts with applications in Section 7 that tie them together. Sections 2-5 develop the theoretical comparative statics results: Section 2 covers mathematical preliminaries, Section 3 presents the model, and Sections 4 and 5 provide conditions on the inverted and noninverted Jacobian, respectively. The second part, Section 6, studies the empirical identification of MATTS. Section 8 concludes.

2 Mathematical Preliminaries

Say that the real variable x is positive if $x \ge 0$ and strictly positive if x > 0. Similarly, x is negative if $x \le 0$ and strictly negative if x < 0.

Consider the $n \times n$ real matrix $A = (a_{ij})$. A is a *P*-matrix (P_0 -matrix) if all of its principal minors are strictly positive (positive). A is a *Z*-matrix if all of its off-diagonal terms are negative. A is an *M*-matrix if it is a nonsingular *Z*-matrix and it has a positive inverse, $A^{-1} \ge 0$. A is strictly diagonally dominant (SDD) if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$
 for $i = 1, ..., n$.

The diagonal entry of an SDD matrix is larger in magnitude than the sum of the absolute values of the elements in the same row.

Define

$$r_i^+ = \max\{0, a_{ij} | j \neq i\}$$
 and $c_j^+ = \max\{0, a_{ij} | i \neq j\}$

as the largest nonnegative element in row i (column j). If all elements in the row

⁴See also Hoffman (1965).

(column) are strictly negative, set r_i^+ (c_j^+) equal to zero. A is a *B*-matrix if

$$\sum_{j=1}^{n} a_{ij} > nr_i^+.$$
 (2.1)

for i = 1, ..., n. A is a *B*-matrix by columns if, for j = 1, ..., n,

$$\sum_{i=1}^{n} a_{ij} > nc_j^+.$$
 (2.2)

For each row of a *B*-matrix, the average entry is positive and greater than each of the off-diagonal entries. A useful mnemonic is that *B*-matrices are mean positive dominant. If inequalities (2.1) and (2.2) are weak, then A is a B_0 -matrix and B_0 -matrix by columns, respectively.

An *SDD* matrix with a strictly positive diagonal is a *P*-matrix with a strictly positive determinant. A *B*-matrix (B_0 -matrix) also has a strictly positive (positive) determinant, and no weaker linear condition exists under which *A* has this property (Carnicer et al., 1999). A *B*-matrix (B_0 -matrix) is also a *P*-matrix (P_0 -matrix) (Peña, 2001). A *B*-matrix (B_0 -matrix) has a strictly positive (positive) diagonal; in fact, $a_{ii} > r_i^+$ ($a_{ii} \ge r_i^+$) for all *i* (Peña, 2001).⁵ These and other properties of *B*-matrices are useful in proving existence, uniqueness, and stability of equilibria (Christensen, 2019).

3 The Comparative Statics Problem

The following framework is standard, except for (1) I adopt the language of reduced form causal estimation to describe the problem and (2) I introduce a novel approach to accommodate different treatment groups. The goal is to provide comparative statics predictions that are valid for any treatment group and for any treatment that has a positive direct impact on outcomes. In this way, the validity of any given statistical test of the sign restriction will not depend on the identity of the treatment group or whether treated units receive the treatment in the prescribed amount.

The population of n units is partitioned into the treated $(g_i = 1)$ and untreated $(g_i = 0)$. Call $g = (g_1, ..., g_n)$ the group assignment vector, or simply the assignment.

⁵For a *B*-matrix, $a_{ii} > nr_i^+ - \sum_{j \neq i} a_{ij} \ge nr_i^+ - (n-1)r_i^+ = r_i^+ \ge 0$. For a *B*₀-matrix the first inequality is weak, but otherwise the argument is the same.

A unit is in the treated group if it receives a shock—or treatment—in the sense that a parameter changes which directly affects its optimal decision.

There are *n* continuously differentiable functions $f^i(y; \lambda)$ for i = 1, ..., n, where $y = (y_1, ..., y_n)$ is a vector of endogenous outcome variables and $\lambda = (\lambda_1, ..., \lambda_n)$ is a vector of exogenous parameters. Assume $y \in \mathcal{Y} \subset \mathbb{R}^n$ and $\lambda \in \Lambda \subset \mathbb{R}^n$, where \mathcal{Y} and Λ are open sets. As an example, each function f^i may represent the marginal profit of firm i, y_i its output, and λ_i the unit tax applied to firm i. The equation $f^i(y; \lambda) = 0$ would be firm i's first order condition for profit maximization.

Given $\lambda = \overline{\lambda}$, an equilibrium \overline{y} is a solution to the system of equations $f^i(y; \overline{\lambda}) = 0$ for i = 1, ..., n. More compactly, let $f = (f^1, ..., f^n)$. Then, in equilibrium, $f(\overline{y}; \overline{\lambda}) = 0$.

To describe the equilibrium effect of a change in the parameters λ , we need a way to select which parameters are changing to accommodate different treatment groups. To simplify, assume $\frac{\partial f^i}{\partial \lambda_j} = 0 \forall j \neq i$. This means, for example, that firm *i*'s output is directly affected by a tax on firm *i*, but not by a tax on firm *j*.⁶ Letting *I* denote the $n \times n$ identity matrix, put G = Ig as the diagonal matrix whose main diagonal is the group assignment vector. Let $D_{\lambda}f(\bar{y};\bar{\lambda})$ be the $n \times 1$ vector with typical element $f_{\lambda}^i \equiv \frac{\partial f^i}{\partial \lambda_i}$. Then $GD_{\lambda}f(\bar{y};\bar{\lambda})$ is the $n \times 1$ vector of direct treatment effects.

The Jacobian of f, $D_y f(y; \lambda)$, is the $n \times n$ matrix of partial derivatives, $f_j^i \equiv \frac{\partial f^i}{\partial y_j}$. Let $Dy(\bar{\lambda})$ denote the vector of equilibrium treatment effects whose i^{th} element is $\frac{d\bar{y}_i}{d\lambda} \equiv \sum_{j:g_j=1} \frac{d\bar{y}_i}{d\lambda_j}$. Then by the IFT, if $D_y f(\bar{y}; \bar{\lambda})$ is nonsingular,

$$Dy(\bar{\lambda}) = -[D_y f(\bar{y}; \bar{\lambda})]^{-1} G D_{\lambda} f(\bar{y}; \bar{\lambda}).$$
(3.1)

I can now define MATTS, ATTS, and ASU. Let n_t and n_u be the number of units in the treated and untreated groups, respectively, in the population. Note that $n_t + n_u = n$. Let δ_{ij} be the typical element of $-[D_y f(\bar{y}; \bar{\lambda})]^{-1}$. Then from (3.1) it follows that the average effect of treatment on the treated with spillovers (ATTS)

⁶Firm *i* is *indirectly* affected by the tax on firm *j* if $\frac{\partial f^i}{\partial y_j} \neq 0$.

and the average spillover effect on the untreated (ASU) are, respectively,

$$ATTS = \frac{1}{n_t} \sum_{r:g_r=1} \sum_{s:g_s=1} \delta_{rs} f^s_{\lambda} \text{ and}$$
(3.2)

$$ASU = \begin{cases} \frac{1}{n_u} \sum_{r:g_r=0} \sum_{s:g_s=1} \delta_{rs} f_{\lambda}^s & \text{if } n_t < n\\ 0 & \text{if } n_t = n. \end{cases}$$
(3.3)

If $n_t = n$, then the whole population is treated so ASU is conceptually undefined, but for technical reasons I set it equal to zero. Then if $n_t < n$,

$$MATTS = ATTS - ASU$$
$$= \frac{1}{n_t} \sum_{r:g_r=1} \sum_{s:g_s=1} \delta_{rs} f^s_{\lambda} - \frac{1}{n_u} \sum_{r:g_r=0} \sum_{s:g_s=1} \delta_{rs} f^s_{\lambda}.$$
(3.4)

And if $n_t = n$, MATTS = ATTS.

Example 1. Let n = 3. By (3.1), equilibrium treatment effects are

$$\begin{bmatrix} \frac{d\bar{y}_1}{d\lambda} \\ \frac{d\bar{y}_2}{d\lambda} \\ \frac{d\bar{y}_3}{d\lambda} \end{bmatrix} = \begin{bmatrix} \delta_{11}f_{\lambda}^1g_1 + \delta_{12}f_{\lambda}^2g_2 + \delta_{13}f_{\lambda}^3g_3 \\ \delta_{21}f_{\lambda}^1g_1 + \delta_{22}f_{\lambda}^2g_2 + \delta_{23}f_{\lambda}^3g_3 \\ \delta_{31}f_{\lambda}^1g_1 + \delta_{32}f_{\lambda}^2g_2 + \delta_{33}f_{\lambda}^3g_3 \end{bmatrix}.$$

Suppose only units 1 and 2 are treated, or g = (1, 1, 0). Then

$$ATTS = \frac{1}{2} \left\{ \left(\delta_{11} f_{\lambda}^{1} + \delta_{12} f_{\lambda}^{2} \right) + \left(\delta_{21} f_{\lambda}^{1} + \delta_{22} f_{\lambda}^{2} \right) \right\}$$
$$= \frac{1}{2} \left\{ \left(\delta_{11} + \delta_{21} \right) f_{\lambda}^{1} + \left(\delta_{22} + \delta_{12} \right) f_{\lambda}^{2} \right\},$$
$$ASU = \delta_{31} f_{\lambda}^{1} + \delta_{32} f_{\lambda}^{2},$$

and

$$MATTS = \left[\frac{1}{2}\left(\delta_{11} + \delta_{21}\right) - \delta_{31}\right]f_{\lambda}^{1} + \left[\frac{1}{2}\left(\delta_{22} + \delta_{12}\right) - \delta_{32}\right]f_{\lambda}^{2}.$$

Analogous formulas can be written for the other non-zero assignment vectors: (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 0, 1), (0, 1, 1), and (1, 1, 1).

4 Conditions on Equilibrium Spillover Effects

I begin with conditions on the elements of the negated Jacobian inverse, δ_{ij} , which determine the sign of MATTS and other quantities of interest.

The first result can be illustrated in the n = 3 case from Example 1. When $g = (1, 1, 0), ATTS \ge 0$ whenever $f_{\lambda}^1, f_{\lambda}^2 \ge 0$ if and only if $\delta_{11} + \delta_{21} \ge 0$ and $\delta_{21} + \delta_{22} \ge 0$. Note that the latter two inequalities involve the partial sums of columns 1 and 2 of the negated Jacobian inverse. Call these *diagonally-centered partial column* sums because each one includes its diagonal term, δ_{11} and δ_{22} .

Similarly, $ASU \ge 0$ whenever $f_{\lambda}^1, f_{\lambda}^2 \ge 0$ iff $\delta_{31} \ge 0$ and $\delta_{32} \ge 0$. The latter terms are the elements that were excluded from the diagonally-centered partial column sums of columns 1 and 2. Call these non-diagonally-centered partial column sums.

Finally, $MATTS \ge 0$ whenever $f_{\lambda}^1, f_{\lambda}^2 \ge 0$ if the average of the terms included in the diagonally-centered partial column sum exceeds the average of the terms excluded from that sum, by column. That is, $\frac{1}{2}(\delta_{11} + \delta_{21}) \ge \delta_{31}$ and $\frac{1}{2}(\delta_{12} + \delta_{22}) \ge \delta_{32}$.

Theorem 1 generalizes this argument to allow for any group assignment and any finite n. Define $\Gamma = \{g = (g_1, ..., g_n) \in \mathbb{R}^n | g_i \in \{0, 1\} \forall i \text{ and } g_i = 1 \text{ for some } i\}$ as the set of all possible non-zero group assignment vectors. I suppress arguments in the following results, but they are all equilibrium results.

Theorem 1. Suppose $D_y f(\bar{y}; \lambda)$ is nonsingular.

- 1. $ATTS \ge 0$ for any $g \in \Gamma$ whenever $D_{\lambda}f \ge 0$ iff $\sum_{r:g_r=1} \delta_{rs} \ge 0$ for all $s:g_s = 1$ and all $g \in \Gamma$. Moreover ATTS > 0 for any $g \in \Gamma$ whenever $D_{\lambda}f > 0$ iff, for any $g \in \Gamma$, $\sum_{r:g_r=1} \delta_{rs} \ge 0$ for all $s:g_s = 1$ with strict inequality for some $s:g_s = 1$.
- 2. $ASU \ge 0$ for any $g \in \Gamma$ whenever $D_{\lambda}f \ge 0$ iff $\sum_{r:g_r=0} \delta_{rs} \ge 0$ for all $s:g_s = 1$ and all $g \in \Gamma$. Moreover, if $n_t < n$, then ASU > 0 for any $g \in \Gamma$ whenever $D_{\lambda}f > 0$ iff, for any $g \in \Gamma$, $\sum_{r:g_r=0} \delta_{rs} \ge 0$ for all $s:g_s = 1$ with strict inequality for some $s:g_s = 1$.
- 3. $MATTS \ge 0$ for any $g \in \Gamma$ whenever $D_{\lambda}f \ge 0$ iff $\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \ge \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}$ for all $s:g_s = 1$ and all $g \in \Gamma$ such that $n_t < n$, and $\frac{1}{n} \sum_{r=1}^n \delta_{rs} \ge 0$ for s = 1, ..., n. Moreover, MATTS > 0 for any $g \in \Gamma$ whenever $D_{\lambda}f > 0$ iff, for any $g \in \Gamma$, $\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \ge \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}$ for all $s:g_s = 1$ (with strict inequality for some $s:g_s = 1$) and $\frac{1}{n} \sum_{r=1}^n \delta_{rs} \ge 0$ for s = 1, ..., n (with strict inequality for some s = 1, ..., n).

Remark 1. Variations of this result are straightforward. In addition to the necessary and sufficient result in part 1, one can also write

- $ATTS \ge 0$ for any group assignment vector $g \in \Gamma$ whenever $D_{\lambda}f \le 0$ iff $\sum_{r:g_r=1} \delta_{rs} \le 0$ for all $s:g_s = 1$ and all $g \in \Gamma$.
- $ATTS \leq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_{\lambda}f \leq (\geq)0$ iff $\sum_{r:g_r=1} \delta_{rs} \geq (\leq)0$ for all $s:g_s = 1$ and all $g \in \Gamma$.

Conditions for ATTS > (<)0 can be generalized analogously. Similar variations hold for parts 2 and 3 of the theorem.

In words, Theorem 1 says that ATTS (ASU) is positive for any non-trivial assignment whenever direct treatment effects are positive if, and only if, every (non-)diagonallycentered partial column sum of the negated Jacobian inverse is positive. Moreover, ATTS (ASU) is strictly positive iff every (non-)diagonally-centered partial column sum of the negated Jacobian inverse is positive, with at least one sum being strictly positive for every nontrivial assignment.

Theorem 1 also says that MATTS is positive iff the average of any set of column entries which includes the diagonal term is larger than the average of the remaining column entries. This is harder to conceptualize, but there is a profitable simplification.

I illustrate the idea in the n = 3 case. Writing out the inequalities in part 3 of Theorem 1, we have, for each i = 1, 2, 3 and $j \neq k \neq i$:

$$\delta_{ii} \ge \frac{1}{2} \left(\delta_{ji} + \delta_{ki} \right) \tag{4.1}$$

$$\frac{1}{2}\left(\delta_{ii}+\delta_{ji}\right) \ge \delta_{ki} \tag{4.2}$$

$$\frac{1}{2}\left(\delta_{ii}+\delta_{ki}\right) \ge \delta_{ji} \tag{4.3}$$

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \ge 0. \tag{4.4}$$

Those four inequalities are equivalent to these three:

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \ge 3\delta_{ji} \tag{4.5}$$

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \ge 3\delta_{ki} \tag{4.6}$$

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \ge 0. \tag{4.7}$$

Inequalities (4.5)-(4.7) are the same as (4.2)-(4.4). To get (4.1), add (4.5) and (4.6), and then rearrange:

$$2(\delta_{ii} + \delta_{ji} + \delta_{ki})\delta_{ii} \ge 3(\delta_{ji} + \delta_{ki}) \Longrightarrow \delta_{ii} \ge \frac{1}{2}(\delta_{ji} + \delta_{ki})$$

Notice that inequalities (4.5)-(4.7) are equivalent to saying that the negated Jacobian inverse, $-[D_{\lambda}f]^{-1}$, is a B_0 -matrix by columns! The next result says that this result holds for arbitrary $n < \infty$.

Theorem 2. Suppose $D_y f(\bar{y}; \bar{\lambda})$ is nonsingular. $MATTS \ge (\le)0$ for any assignment $g \in \Gamma$ whenever $D_{\lambda}f \ge (\le)0$ iff $-[D_y f]^{-1}$ is a B_0 -matrix by columns. Moreover, MATTS > (<)0 for any assignment $g \in \Gamma$ whenever $D_{\lambda}f > (<)0$ if $-[D_y f]^{-1}$ is a B-matrix by columns.

From a linear algebra perspective, this result says that for a given matrix, every column sum is positive and the average of any set of column entries which includes the diagonal term is larger than the average of the remaining column entries iff the matrix is a B_0 -matrix by columns. This characterization of *B*-matrices is new.

To interpret this result, think of the elements of the negated Jacobian inverse, δ_{ij} , as the equilibrium effect of a one unit increase in unit *i*'s outcome on unit *j*'s outcome for $i \neq j$. If the negated Jacobian is a B_0 -matrix by columns, this means that the equilibrium effect of a change in any unit *i*'s outcome on unit *j*'s outcome cannot be larger than the average equilibrium effect on unit *j*, where the average equilibrium effect includes the own equilibrium effect δ_{jj} . This condition rules out direct spillover effects which accumulate into outlier equilibrium spillover effects.

Theorem 2 is a fascinating result. First it identifies a well-known class of matrices, B-matrices, which characterize the negated Jacobian inverse such that MATTS is positive. Thus, to determine the type of direct spillover effects under which MATTS is (strictly) positive, we can focus attention on the class of matrices whose transposed inverse is a B_0 -matrix (B-matrix). This task is taken up in the next section.

In addition, the characterization of MATTS provided in part 3 of Theorem 1 relies on significantly more inequalities than the characterization in Theorem 2. For a given n, part 3 of Theorem 1 requires one to check

$$\sum_{j=0}^{n-1} \left(\begin{array}{c} n-1\\ j \end{array} \right)$$

inequalities per column while the B_0 -matrix property requires only n.⁷ The remaining inequalities are redundant. To give a sense of the scale of simplification, if n = 15, then part 3 of Theorem 1 involves checking 16,384 inequalities per column for a total of 245,760 inequalities; Theorem 2 reduces this to 15 per column for a total of 225. If n = 25, the totals are over 419 million compared to just 625.

Finally, three useful corollaries easily obtain as a consequence of the B_0 -matrix characterization. First, if MATTS is always positive, then ATTS is also always positive. On first impression, this is surprising since MATTS is the difference between ATTS and ASU. But the *B*-matrix condition implies that the own equilibrium effect dominates negative equilibrium effects from other units. Second, treatment increases the total, and thus average, outcome. That is, if $Y \equiv \sum_{i=1}^{n} \frac{d\bar{y}_i}{d\lambda} \geq 0$. Third, if there is a single treated unit, then the impact of treatment on this unit is positive and larger in magnitude than the impact on any other unit.

Corollary 1. Suppose $-D_y f$ is nonsingular. If $-[D_y f]^{-1}$ is a *B*-matrix (B_0 -matrix) by columns, then

- 1. $ATTS > (\geq) 0$ for every $g \in \Gamma$ whenever $D_{\lambda}f > 0$,
- 2. $\frac{d\bar{Y}}{d\lambda} > (\geq) \ 0$ for every $g \in \Gamma$ whenever $D_{\lambda}f > 0$ and
- 3. if, in addition, unit i is the only treated unit $(n_t = 1), \frac{d\bar{y}_i}{d\lambda} > (\geq) \left| \frac{d\bar{y}_j}{d\lambda} \right|$ for all $j \neq i$ whenever $D_{\lambda}f > 0$.

Proof. I prove the result for *B*-matrices. The result for B_0 -matrices is analogous.

(1) By the definition of a *B*-matrix by columns, $\sum_{i} \delta_{ij} > nc_{j}^{+}$ for j = 1, ..., n. Let $H = \{h \in \mathbb{N} | 1 \le h \le n \text{ and } \delta_{hj} < 0\}$. If $g_s = 1$, then $\sum_{r:g_r=1} \delta_{rs} \le 0$ only if there are some terms δ_{rs} in the sum such that $r \in H$. But by Proposition 2.4 in Peña (2001), $\delta_{jj} > \sum_{h \in H} |\delta_{hj}|$.⁸ It follows that for all $g \in \Gamma$, $\sum_{r:g_r=1} \delta_{rs} > 0$ for all $s:g_s = 1$. Thus, ATTS > 0 by Theorem 1.

⁷For each column, the inequalities in part (3) of Theorem 1 involve every difference between the diagonally-centered partial column sum and the sum of the remaining column entries. Thus, the number of inequalities to check is the same as the number diagonally-centered partial column sums. Each of these sums includes the diagonal element, to which we add between 0 and n-1 off-diagonal elements. If j off-diagonal elements are included, there are n-1 choose j inequalities.

 $^{{}^{8}\}delta_{jj} > nc_{j}^{+} - \sum_{i \neq j} \delta_{ij} = nc_{j}^{+} - \sum_{h \neq j, h \neq H} \delta_{hj} + \sum_{h \in H} |\delta_{hj}|.$

(2) By equation (3.1), $\frac{d\bar{Y}}{d\lambda} = \sum_i \sum_j \delta_{ij} f^j_{\lambda} g_j = \sum_j (\sum_i \delta_{ij}) f^j_{\lambda} g_j$. Note that $\sum_i \delta_{ij} > 0$ because $-[D_y f]^{-1}$ is a *B*-matrix by columns. This proves the result.

(3) If $g_i = 1$ and $g_j = 0$ for all $j \neq i$, then by equation (3.1) we have $\frac{d\bar{y}_i}{d\lambda} = \delta_{ii} f_{\lambda}^i$ and $\frac{d\bar{y}_j}{d\lambda} = \delta_{ji} f_{\lambda}^j$ for all $j \neq i$, so it suffices to prove $\delta_{ii} > |\delta_{ji}|$ for all $j \neq i$. But this follows from Proposition 2.4 in Peña (2001).

5 Conditions on Direct Spillover Effects

Theorems 1-2 are especially useful if the Jacobian can be inverted in closed form, but often this is infeasible. This section gives conditions on the (noninverted) Jacobian under which MATTS is positive. Put another way, rather than finding conditions on the equilibrium spillover effects, δ_{ij} , I now focus on finding conditions on the direct spillover effects, f_i^i .

By Theorem 2 and Corollary 1 it is sufficient to find conditions on the negated Jacobian such that its inverse is a *B*-matrix by columns. This is a challenging problem in general, but I make headway in some economically relevant special cases. Similar to Christensen (2019), the overall theme of the findings is that a trade-off exists between the heterogeneity and size of spillovers.

5.1 Anonymous-By-Unit Spillovers

A great deal of structure emerges if spillover effects are *anonymous-by-unit*, or $f_i^i = \alpha_i$ and $f_j^i = \beta_i$ for all $j \neq i$ and all i = 1, ..., n. In this case, a unit increase in y_j has the same effect on y_i as a unit increase in y_k , for any $j \neq k \neq i$. In matrix form,

$$D_y f(\bar{y}, \bar{\lambda}) = \begin{bmatrix} \alpha_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & \alpha_2 & \cdots & \beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_n & \cdots & \alpha_n \end{bmatrix}.$$

This case arises in two important classes of models: (1) when the component functions take the form $f^i(y; \lambda) = f^i(y_i, \sum_{j \neq i} y_j; \lambda)$ such as in Cournot competition where a firm's demand, which can be different for each firm, depends on rivals' outputs only through their sum, and (2) at symmetric equilibria when the component functions f^i are symmetric in y_{-i} . See Section 7 for more detail and examples. Dixit (1986) derives closed form comparative statics formulae for this case. To sign the comparative statics, he assumes that $\alpha_i < 0$ and the matrix $-D_y f(\bar{y}, \bar{\lambda})$ is SDD: for $i = 1, ..., n, |-\alpha_i| > (n-1)|-\beta_i|$, or,

$$\begin{cases} \alpha_i < (n-1)\beta_i & \text{if } \beta_i \le 0\\ \alpha_i < -(n-1)\beta_i & \text{if } \beta_i > 0. \end{cases}$$
(5.1)

Dixit obtains the following three results on equilibrium variables for the case when only unit *i* is treated $(n_t = 1)$. If $f_{\lambda}^i > 0$, then the treated unit's outcome increases $(d\bar{y}_i/d\lambda > 0)$, the total outcome increases $(d\bar{Y}/d\lambda > 0)$, and the outcome of untreated units $j \neq i$ increases or decreases as β_j is negative or positive.

Christensen (2019) showed that the first two results hold under the weaker assumption that $-D_y f(\bar{y}, \bar{\lambda})$ is a *B*-matrix, that is, when $-\alpha_i - (n-1)\beta_i > n \max\{0, -\beta_i\}$ for i = 1, ..., n, or

$$\begin{cases} \alpha_i < \beta_i & \text{if } \beta_i \le 0\\ \alpha_i < -(n-1)\beta_i & \text{if } \beta_i > 0. \end{cases}$$
(5.2)

To see that this is a weaker restriction than SDD, note that (5.1) implies (5.2). On the other hand, here is *B*-matrix that is not SDD:

Remarkably, by direct computation we can also sign MATTS, ATTS, and ASU for any assignment, not just those with a single treated unit. To this end, let

$$\Gamma = 1 + \sum_{i=1}^{n} \beta_i / (\alpha_i - \beta_i).$$

Per equation (41.1) in Dixit (1986) we have

$$\frac{d\bar{y}_i}{d\lambda} = -\frac{g_i f_\lambda^i}{\alpha_i - \beta_i} + \frac{\beta_i}{\Gamma(\alpha_i - \beta_i)} \sum_{j=1}^n \frac{g_j f_\lambda^j}{\alpha_j - \beta_j} \text{ and}$$
(5.3)

$$\frac{dY}{d\lambda} = -\frac{1}{\Gamma} \sum_{i=1}^{n} \frac{g_i f_{\lambda}^i}{\alpha_i - \beta_i}.$$
(5.4)

If $-D_y f$ is a *B*-matrix and $\beta_i > 0$ then

$$-\frac{1}{n} = -\frac{\beta_i}{n\beta_i} < \frac{\beta_i}{\alpha_i - \beta_i} < 0.$$
(5.5)

If $\beta_i \leq 0$ then $\beta_i/(\alpha_i - \beta_i) \geq 0$. From these two facts we have $\Gamma = 1 + \sum_{i=1}^n \frac{\beta_i}{\alpha_i - \beta_i} > 0$. It follows that $\frac{d\bar{Y}}{d\lambda} > 0$ for any assignment $g \in \Gamma$ whenever $D_{\lambda}f > 0$.

Turning to average impacts, note that

$$\begin{split} ATTS &= \frac{1}{n_t} \left[-\sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j} + \sum_{k:g_k=1} \frac{\beta_k / (\alpha_k - \beta_k)}{\Gamma} \sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j} \right] \\ &= -\frac{1}{n_t} \left[\sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j} \left(1 - \sum_{k:g_k=1} \frac{\beta_k / (\alpha_k - \beta_k)}{\Gamma} \right) \right], \\ ASU &= \frac{1}{n_u} \sum_{k:g_k=0} \frac{\beta_k / (\alpha_k - \beta_k)}{\Gamma} \sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j}, \text{ and} \\ MATTS &= -\sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j} \left\{ \begin{array}{c} \frac{1}{n_t} \left(1 - \sum_{k:g_k=1} \frac{\beta_k / (\alpha_k - \beta_k)}{\Gamma} \right) \\ &+ \frac{1}{n_u} \sum_{k:g_k=0} \frac{\beta_k / (\alpha_k - \beta_k)}{\Gamma} \right\}. \end{split}$$

Because

$$\Gamma - \sum_{k:g_k=1} \frac{\beta_k}{\alpha_k - \beta_k} = 1 + \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} > 1 - \frac{n_u}{n} \ge 0,$$

it follows that ATTS> 0. The sign of ASU is ambiguous in general, but it is strictly positive (strictly negative) if $\beta_k \ge (\le) 0$ for all $k : g_k = 0$ with strict inequality for some $k : g_k = 0$. To sign MATTS, observe that

$$\begin{split} \Gamma - \sum_{j:g_j=1} \frac{\beta_j}{\alpha_j - \beta_j} + \frac{n_t}{n_u} \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} &= 1 + \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} + \frac{n_t}{n_u} \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} \\ &= 1 + \frac{n}{n_u} \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} \\ &> 1 - \frac{n}{n_u} \frac{n_u}{n} \\ &= 0. \end{split}$$

It follows that MATTS > 0. These findings are summarized in the theorem below.

Theorem 3. Suppose spillovers are anonymous-by-unit and that $-D_y f(\bar{y}, \bar{\lambda})$ is a *B*-matrix. Then for any $g \in \Gamma$ and any vector of direct treatment effects $D_{\lambda}f > 0$,

- 1. MATTS > 0,
- 2. Corollary 1 applies, and

3.
$$\begin{cases} ASU > 0 & \text{if } \beta_k \ge 0 \text{ for all } k : g_k = 0 \text{ and } \beta_k > 0 \text{ for some } k : g_k = 0 \\ ASU < 0 & \text{if } \beta_k \le 0 \text{ for all } k : g_k = 0 \text{ and } \beta_k > 0 \text{ for some } k : g_k = 0. \end{cases}$$

Is there any weaker condition which guarantees that MATTS is strictly positive when spillovers are anonymous-by-unit? No, not if $\alpha_i \neq \beta_i$. In this case $-D_y f$ is a *B*-matrix if it is a B_0 -matrix, and $-D_y f$ is a *B*-matrix if and only if $[-D_y f]^{-1}$ is a *B*-matrix by columns. This is a new result for *B*-matrices.

Lemma 1. Suppose for all i = 1, ..., n, $f_i^i = \alpha_i \neq 0$ and $f_j^i = \beta_i \neq \alpha_i$ for all $j \neq i$. Then $-D_y f(\bar{y}; \bar{\lambda})$ is a *B*-matrix (by rows) if and only if $-[D_y f(\bar{y}; \bar{\lambda})]^{-1}$ a *B*-matrix by columns.

Sufficiency follows from Theorem 2 since we have shown that MATTS>0 when $-D_y f$ is a *B*-matrix. I prove necessity and provide an alternate sufficiency proof in the Appendix. Lemma 1 and Theorem 2 imply the following:

Corollary 2. If spillover effects are anonymous-by-unit and $\alpha_i \neq \beta_i \; \forall i$, then MATTS> 0 for any $g \in \Gamma$ whenever $D_{\lambda}f > 0$ if and only if $-D_yf(\bar{y}, \bar{\lambda})$ is a B-matrix.

5.2 **Positive Spillovers**

Assume spillovers are positive, or $f_j^i \ge 0$ for all $i \ne j$. If, in addition, $-D_y f$ has a strictly positive diagonal then it is a Z-matrix. If it is also an *M*-matrix it has a positive inverse, and thus the outcome of every unit weakly increases; hence, ATTS and ASU are positive. In fact, this is also a necessary condition. A stronger condition is needed to ensure that MATTS is positive.

Theorem 4. Suppose for i = 1, ..., n, $f_j^i \ge 0$ for all $i \ne j$ and $f_i^i < 0$.

1. For all i = 1, ..., n, $\frac{d\bar{y}_i}{d\lambda} \ge 0$ for any assignment $g \in \Gamma$ whenever $D_{\lambda}f \ge 0$ if and only if $-D_y f$ is an *M*-matrix. It follows that $ATTS \ge 0$ and $ASU \ge 0$.

2. Suppose

$$-f_i^i > (n-1) \max_{j \neq i} \{f_j^i\} \quad \forall i.$$
(5.6)

Then $-D_y f$ is an *M*-matrix and $[-D_y f]^{-1}$ is a *B*-matrix by columns. Thus, part (1) applies, MATTS > 0, and the results of Corollary 1 apply.

Condition (5.6) implies that $-D_y f$ is an *M*-matrix and a *B*-matrix. To see this, note that $r_i^+ = \max\{0, -f_j^i | j \neq i\} = 0$. Then, because $(n-1) \max_{j\neq i} \{f_j^i\} \ge \sum_{j\neq i} f_j^i$, it follows that $\sum_{j=1}^n -f_j^i > nr_i^+ = 0$. Thus, $-D_y f$ is a *B*-matrix. A *B*-matrix is a *P*matrix, so it follows that $-D_y f$ is also an *M*-matrix because a nonsingular *Z*-matrix that is also a *P*-matrix is an *M*-matrix (Plemmons, 1977). Thus, part 2 of Theorem 4 identifies a subclass of *B*-matrices whose inverse is a *B*-matrix by columns. Given the sign restrictions on f_j^i , (5.6) cannot be weakened while retaining the property that $-D_y f$ is a *B*-matrix because it reduces to the definition of a *B*-matrix if n = 2.

Intuitively, when spillovers are positive, an increase in any unit's outcome (weakly) increases the outcomes of all other units. Thus, a treatment that increases the outcome of any unit(s) should increase the outcome of all units, provided that the equilibrium system is well-behaved. Consequently, we expect ATTS and ASU to be positive. The system is well-behaved if its Jacobian is an *M*-matrix, which is a type of stability requirement (Plemmons, 1977; Christensen and Cornwell, 2018).

To see that signing MATTS requires stronger conditions, observe that

$$-D_y f = \begin{bmatrix} 1 & -1.1 \\ 0 & 1 \end{bmatrix} \text{ has inverse } [-D_y f]^{-1} = \begin{bmatrix} 1 & 1.1 \\ 0 & 1 \end{bmatrix},$$

so $-D_y f$ is an *M*-matrix but $[-D_y f]^{-1}$ is not a *B*-matrix by columns. The latter implies by Theorem 2 that there exists a vector of strictly positive direct treatment effects and an assignment such that MATTS is strictly negative. This occurs if direct treatment effects are all one, $D_\lambda f = (1, 1)^T$, and only unit 2 is treated, g = (0, 1). In this case, $\frac{d\bar{y}_1}{d\lambda} = 1.1$ and $\frac{d\bar{y}_2}{d\lambda} = 1$, but MATTS = 1 - 1.1 = -0.1 < 0.

5.3 Strictly Negative Spillovers

Strong results are also available if spillovers are strictly negative, meaning $f_j^i < 0$ for all i, j. For then the negated Jacobian $-D_y f$ is a strictly positive matrix. If its inverse is an *M*-matrix, the terms on the main diagonal are strictly positive and the off-diagonal terms are negative. From the latter it follows that ASU is negative. In fact, $\frac{d\bar{y}_i}{d\lambda} \leq 0$ for any untreated unit. If, in addition, the inverse is a *B*-matrix by columns, it follows that ATTS and MATTS are strictly positive, as claimed in Theorem 5 below. In fact, $\frac{d\bar{y}_i}{d\lambda} > 0$ for any treated unit if treatment is uniform, or $f_{\lambda}^i = \bar{f}_{\lambda} > 0 \forall i$. The result relies largely on Willoughby (1977) which provides tight (for $n \geq 4$) sufficient conditions under which the inverse of a positive matrix is an *M*-matrix.

Lemma 2 (Willoughby, 1977). Suppose $-f_j^i > 0$ for all i, j = 1, ..., n. Assume $0 < m \le M < 1$ and for $i \ne j$, $0 < m \le \frac{f_j^i}{f_i^i} \le M < 1$. Let the interpolation parameter, s, be defined by

$$M^2 = sm + (1 - s)m^2.$$

Further suppose that any of the following conditions is satisfied:

- 1. n = 2,
- 2. m = M, or
- 3. $n \geq 3$ and $s \leq \frac{1}{n-2}$.

Then $[-D_y f]^{-1}$ exists and is a SDD (by rows and columns) M-matrix.

Theorem 5. Suppose any of the conditions of Lemma 2 are satisfied. Then for any group assignment $g \in \Gamma$ and $D_{\lambda}f > 0$, MATTS > 0, $ASU \leq 0$, and $\frac{d\bar{y}_i}{d\lambda} \leq 0$ for any $i : g_i = 0$. Corollary 1 also applies. If, in addition, $f_{\lambda}^i = \bar{f}_{\lambda} > 0 \,\forall i$, then $\frac{d\bar{y}_i}{d\lambda} > 0$ for any $i : g_i = 1$.

While Lemma 2 concludes that $[-D_y f(\bar{y}; \bar{\lambda})]^{-1}$ is a *SDD* matrix by rows and columns, because it is also an *M*-matrix, this is equivalent to saying that it is a *B*-matrix by rows and columns. The reason for this is that $r_i^+ = c_j^+ = 0$, so the *B*-matrix definition requires only that its row and column sums are positive. For each column j, this means, $a_{jj} > -\sum_{i \neq j} a_{ij}$, which is equivalent to $|a_{jj}| > \sum_{i \neq j} |a_{ij}|$. And similarly for each row i. Hence, apply Theorem 2 to get Theorem 5.

Intuitively, if a single unit receives a strictly positive treatment, downward pressure is exerted on the outcomes of all other units when direct spillover effects are strictly negative. Consequently, we expect the treated unit's outcome to increase while untreated units' outcomes decrease, and hence MATTS is strictly positive. For a well-behaved system—as defined by the conditions in Lemma 2—this intuition extends to any treatment group after accounting for equilibrium spillovers. Note that these conditions are global in the sense that they constrain the heterogeneity of direct spillover effects of all units jointly. In contrast, Theorems 3 and 4 constrain the heterogeneity of these effects acting on a single unit. Put differently, Lemma 2 is a joint condition on all the off-diagonal terms of the negated Jacobian, whereas the other results are conditions which apply independently to each row of the matrix.

5.4 Small Spillovers

The next result formalizes the intuition that small spillovers should not be able to overcome the direct effect of treatment. Viewed through a linear algebra lens, Theorem 6 identifies a new subclass of B-matrices whose inverse is a B-matrix by columns. The proof relies on an inequality recently established in Norris et al. (2023).

Theorem 6. Consider $-D_y f$. If

$$-f_i^i > (n-1)^2 \sum_{i \neq j} \left| -f_j^i \right| \text{ for } i = 1, ..., n,$$
(5.7)

then

$$\delta_{jj} > (n-1) \sum_{i \neq j} |\delta_{ij}| \text{ for } i = 1, ..., n.$$
 (5.8)

When inequalities (5.7) and (5.8) are satisfied, $-D_y f$ is a SDD (by rows) B-matrix and $-[D_y f]^{-1}$ is a SDD (by columns) B-matrix by columns, respectively. It follows that MATTS > 0 for any $g \in \Gamma$ whenever $D_{\lambda} f > 0$ and that Corollary 1 applies.

If we allow f_j^i to take any sign, then condition (5.7) cannot be weakened while retaining the property that $-D_y f$ is a *B*-matrix because it reduces to the definition of a *B*-matrix for n = 2 and $-f_2^1 > 0$, for example. Also observe that condition (5.7), which applies if spillovers can take any sign, is stronger than condition (5.6), which applies if spillovers are positive. This illustrates that sign heterogeneity in spillover effects contribute to undisciplined comparative statics.

6 Empirical Identification of MATTS

In the empirical literature, *interference* arises when the treatment applied to one unit affects the outcome for other units (Imbens and Rubin, 2015). The standard framework assumes no interference, thereby ruling out spillover effects. In this setting, it is well known that (1) the average treatment effect among the treated (ATT) is identified by the population difference-in-differences (DiD), and (2) if treatment assignment is completely random, the ATT equals the average treatment effect (ATE), which is identified by the population difference-in-means (DiM). This section shows that MATTS generalizes these ATT properties when interference is present.

Another contribution of this section is to highlight the importance of the MATTS estimand, as its natural estimator—the sample DiD—is unbiased from a superpopulation perspective even in the presence of pervasive spillovers. Since Sobel (2006), many studies have noted that estimators unbiased for a given estimand in the absence of interference become biased when interference exists. Butts (2021) and Xu (2023) make a similar point in the DiD context. However, this observation often motivates the development of new causal estimands and, in some cases, new estimators. Many efforts aim to separately identify direct causal effects from spillover effects, which typically require additional assumptions restricting spillovers. Common approaches include partial interference, where some identifiable units remain unaffected by spillovers (e.g., Hudgens and Halloran, 2008), exposure mappings, which reduce the dimensionality of the effective assignment vector (e.g., Manski, 2013), and structural approaches such as linear-in-means models which often require some knowledge of the network structure (e.g., Blume et al., 2015; Kline and Tamer, 2020; Bramoullé et al., 2020).⁹ Basse and Airoldi (2018) show that some restrictions on spillovers are necessary for unbiasedness in an experimental, design-based approach.

In contrast, this paper focuses on MATTS because the sample DiD remains an unbiased estimator (from a superpopulation perspective) even when spillovers are unrestricted.¹⁰ From the perspective of testing comparative statics predictions of a theory, having estimators that remain unbiased under unrestricted spillovers is crucial. As shown in the theoretical sections of this paper, pervasive spillovers can gen-

 $^{^9 {\}rm Other}$ examples include Athey et al. (2018), Vazquez-Bare (2023), Baird et al. (2018), Sävje et al. (2021), Butts (2021), and Xu (2023).

 $^{^{10}}$ Vazquez-Bare (2023) examines randomized experiments in a partial interference setting where spillovers are otherwise unrestricted.

erate counterintuitive comparative statics. If spillovers potentially disrupt theoretical predictions, then using estimators whose unbiasedness relies on restricting spillovers makes it impossible to test this concern. Specifically, if an estimate's sign conflicts with the predicted sign of the estimand, this could stem from spillover-induced bias rather than indicating a flaw in the theory itself.

6.1 DiD with Interference

Let us begin by expanding to allow for interference the potential-outcomes-based presentation of DiD in Roth et al. (2023). Consider a balanced panel with two time periods, t = 1, 2. Units are indexed by i = 1, ..., n. Treated units $(g_i = 1)$ are treated only in period 2, whereas untreated units $(g_i = 0)$ are never treated. Recall that $g = (g_1, ..., g_n) \in \Gamma \bigcup \{0\} \equiv \Gamma'$ is the group assignment vector. In this section, the assignment in which no one is treated, g = 0, is allowed. The potential outcome of unit *i* in period *t* is $Y_{it}(g)$.

Given g, the potential outcomes discrete analogue to MATTS defined in (3.4), is

$$MATTS(g) = \underbrace{\mathbb{E}[Y_{i2}(g) - Y_{i2}(0)|g_i = 1]}_{ATTS(g)} - \underbrace{\mathbb{E}[Y_{i2}(g) - Y_{i2}(0)|g_i = 0]}_{ASU(g)}.$$
 (6.1)

MATTS is the difference between the average effect of treatment among the treated with spillovers (ATTS) and the average spillover effect among the untreated (ASU). ATTS is the period 2 difference between the average outcome among the treated under assignment g and the average potential outcome when no unit is treated. The ASU is the analogous quantity among the untreated. The differential analogues of ATTS and ASU are given in definitions (3.2) and (3.3), respectively.

A key challenge in estimating MATTS, ATTS, or ASU is that the potential outcome in which no one is treated is not observed. To make progress, one typically identifies an observable estimand which is equal to the target estimand. When this is the case, say that the observable estimand *identifies* the unobservable one.

Under parallel trends and no-anticipatory-effects assumptions, the *population DiD*,

$$DiD(g) = \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 0],$$
(6.2)

identifies MATTS. The observable population DiD is the "difference-in-differences" of subpopulation means. Parallel trends asserts that, in the absence of treatment, both groups would have experienced the same outcome evolution, on average. The noanticipatory-effects assumption says that period 1 potential outcomes do not depend on treatment statuses in period 2. In the formal statement of these assumptions below, $g_{-i} = (g_1, ..., g_{i-1}, g_{i+1}, ..., g_n)$ is the treatment statuses of all units but *i*.

(AS1) No anticipatory effects. $Y_{i1}(g_i, g_{-i}) = Y_{i1}(g'_i, g'_{-i})$ for all i and all $g, g' \in \Gamma'$.

(AS2) Parallel Trends.
$$\mathbb{E}[Y_{i2}(0) - Y_{i1}(0)|g_i = 1] = \mathbb{E}[Y_{i2}(0) - Y_{i1}(0)|g_i = 0]$$

These are the analogues to the identification assumptions given in Roth et al. (2023) when interference is allowed. The no-anticipatory-effects assumption (AS1) states that neither a unit's treatment status in period 2 *nor* others' treatment statuses in period 2 have any effect on her outcome in period 1. If there is no interference, then (AS1) reduces to $Y_{i1}(0, g_{-i}) = Y_{i1}(1, g'_{-i})$ for all *i* and any g_{-i}, g'_{-i} .

Although the concepts are named differently, Butts (2021) also observes that the population DiD identifies the difference between ATTS and ASU. In that paper, this observation motivates other estimands of interest which require additional identification assumptions. In contrast, MATTS is the primary estimand of interest here. In addition, I give a proof in the Appendix which clarifies the roles of (AS1) and (AS2), and give an example below which illustrates the necessity of these assumptions.

Suppose, in addition to (AS1)-(AS2), that the period 1 expected outcomes would be the same in the two subpopulations in the absence of treatment:

(AS3) Equal pretreatment expected outcomes. $\mathbb{E}[Y_{i1}(0)|g_i=1] = \mathbb{E}[Y_{i1}(0)|g_i=0].$

Then MATTS is also identified by the observable *population difference-in-means*,

$$DiM(g) = \mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0].$$
(6.3)

Finally, if assignment to treatment is completely random, then the parallel trends and the equal-pretreatment-expected-outcomes assumptions are satisfied. Random assignment means that, for a population of size N, the number of treated units, N_t , is fixed, and N_t units are (as if) randomly selected to receive treatment while the remaining N_u are untreated.

I collect these results in Lemma 3 and Theorem 7. Proofs are in the Appendix.

	Treated	Untreated
Post-Treatment $(t=2)$	3	5
Pre-Treatment $(t = 1)$	9	9

Table 1: Infection Rates (% of population)

Lemma 3. For any treatment group size $n_t \in \{1, ..., n-1\}$, completely random assignment implies parallel trends (AS2) and equal-pretreatment-expected-outcomes (AS3).

Theorem 7.

- 1. Assume (AS1)-(AS2). Then MATTS(g) = DiD(g).
- 2. Assume (AS1)-(AS3). Then MATTS(g) = DiD(g) = DiM(g).
- 3. Assume (AS1) and that assignment is completely random. Then MATTS(g) = DiD(g) = DiM(g).

The intuition behind Theorem 7 can be easily understood through an example. Suppose a vaccine against an infectious disease is administered to a subset of a population. Infection rates are displayed in Table 1. The infection rate is 9 percent in each group before the vaccine is administered. The change in means among the treated is 3 - 9 = -6, while the change in means among the untreated is 5 - 9 = -4. The population DiD is -6 - (-4) = -2. Under the assumptions of no anticipatory effects and parallel trends, the population DiD is interpreted as MATTS—the additional benefit of the vaccine that the treated receive beyond the any herd immunity effect, on average. Notice that the population DiM is 3 - 5 = -2, which is also interpreted as MATTS because the pretreatment infection rates are equal.

If, for some reason, the treated know they will be vaccinated in period 2, and this knowledge causes them to be more lax regarding infection prevention measures in period 1, the pretreatment infection rate among the treated may increase to, say, 10. In this case, the population DiD is (3-10) - (5-9) = -3, but MATTS remains -2. Thus, the no-anticipatory-effects assumption is necessary.

In the absence of treatment, let the period 2 infection rate be $9 + \Delta_t$ and $9 + \Delta_u$ among the treated and untreated groups, respectively. MATTS is $(3-\Delta_t)-(5-\Delta_u) =$ $-2 - (\Delta_t - \Delta_u)$. If infection trends were not parallel, meaning $\Delta_t \neq \Delta_u$, then the infection rate in period 2 could be, for example, 7 ($\Delta_t = -2$) and 9 ($\Delta_u = 0$) if the planned vaccination did not occur, in which case MATTS would be zero, even though the population DiD remains $-2.^{11}$

Finally, if the pretreatment infection rate among the treated were 8 (remaining at 9 among the untreated), the population DiD would be (3-8) - (5-9) = -1, while the population DiM would be -2. In this case, only the population DiD identifies MATTS.

While the focus here is on identification rather than estimation or inference, the estimation is conceptually straightforward from a superpopulation perspective. Assume that treatment is applied to an infinite superpopulation, that both the treated and untreated subsets are infinite, and that the treated and untreated subsets are independently sampled. Then the sample analogues to the population DiD and DiM serve as natural estimators and, by the central limit theorem, have an asymptotically normal distribution as the sample size grows large. The standard two-way fixed effects estimator can be used to estimate MATTS.

6.2 Relationship to the No-Interference Case

Suppose there is no interference:

(AS4) No interference. For all i and $t = 1, 2, Y_{it}(g_i, g_{-i}) = Y_{it}(g_i, g'_{-i})$ for any $g_{-i} \neq g'_{-i}$. Under (AS4), ASU = 0 and ATTS(g) = ATTS(g') for any g, g'. Then

$$ATTS(g) = \mathbb{E}[Y_{i2}(1, g_{-i}) - Y_{i2}(0, g_{-i})|g_i = 1]$$

is the canonical average treatment effect among the treated (ATT). Thus, both ATTS and MATTS generalize the ATT in this sense.

MATTS and ATTS also generalize the canonical average treatment effect (ATE). To see this, first define the *average effect of treatment with spillovers* (AETS) as the average difference between post-treatment outcomes and the potential outcome in which no one is treated in period 2:

$$AETS(g) = \mathbb{E}[Y_{i2}(g) - Y_{i2}(0)].$$

¹¹Note that ATTS and ASU are identified by their natural observable estimands if $\Delta_t = 0$ and $\Delta_t = 0$, respectively. In this case, ATTS is -6 and ASU is -4, but these are strong assumptions.

This is similar to ATTS, except that the expectation is over the whole population, not just the treated. Observe that $AETS(g) = \frac{n_t}{n}ATTS(g) + \frac{n_c}{n}ASU(g)$. Also, if all units are treated, then g is a vector of ones and $AETS(1) = \mathbb{E}[Y_{i2}(1) - Y_{i2}(0)]$.¹²

The canonical ATE is defined without interference. It is independent of g and can be written as

$$ATE = \mathbb{E}[Y_{i2}(1, g_{-i}) - Y_{i2}(0, g'_{-i})] \quad \forall g_{-i}, g'_{-i}.$$

This is equivalent to AETS(1) when there is no interference. Furthermore, under the convention that ASU(1) = 0, we have ATTS(1) = AETS(1) and MATTS(1) = ATTS(1), which implies MATTS(1) = AETS(1).

This discussion demonstrates that MATTS and ATTS generalize ATT and ATE, while AETS also generalizes ATE. In terms of identification, however, MATTS relates to AETS under interference just as ATT relates to ATE in the absence of interference. Specifically, when there is (no) interference, MATTS (ATT) is identified by the population DiD, and AETS (ATE) is identified by the population DiM under completely random assignment.¹³ Moreover, under random assignment, MATTS (ATT) is equal to AETS (ATE) because both equal the population DiM. These statements do not hold for ATTS because the population DiD does not identify it.

Other papers use different terminology for these estimands. Hudgens and Halloran (2008) and Baird et al. (2018) refer to their analogues of the ASU as the indirect causal effect and the spillover on the nontreated, respectively; ATTS as the total causal effect and the intention to treat, respectively; and AETS as the overall and total causal effect, respectively. However, neither paper focuses on MATTS. My goal in introducing these terms is to clarify (1) the relationship between the DiD settings with and without interference and (2) the relationship to random assignment, rather than simply contributing to terminological complexity.

¹²The differential analogue of AETS follows naturally. Because $AETS = \frac{n_t}{n}ATTS + \frac{n_c}{n}ASU$, simply substitute the expressions for ATTS and ASU from equations (3.2) and (3.3). Note that for $g = 1, AETS(1) = \frac{1}{n} \frac{dY(0)}{d\lambda} \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{d\tilde{y}_i(\lambda(0))}{d\lambda}$. ¹³The proof for the AETS is a minor modification of the proof of Theorem 6.1 in Imbens and

¹³The proof for the AETS is a minor modification of the proof of Theorem 6.1 in Imbens and Rubin (2015).

7 Applications

I now consider applications of the comparative statics results to testing the profit maximization hypothesis, oligopoly, contests and symmetric models.

7.1 The Profit Maximization Hypothesis

In the leading example of the first chapter in *The Structure of Economics* (Silberberg, 1978), which expands on ideas introduced in chapters 2 and 3 of Samuelson (1947), the claim is that the profit maximizing assumption can be tested by empirically evaluating whether a firm decreases its output in response to a unit tax. Glossing over some details which are not relevant to the present paper, if we observe that output increases, then we can reject the profit maximizing assumption.

In practice, however, to control for confounding factors, a test of this hypothesis requires a sample of firms that are taxed (the treated group) and a sample that are not taxed (the untreated group). But if a subset of firms is taxed then this will create spillovers in the market through price effects. I give a perverse example in which MATTS is positive even when firms maximize profit. In this case, if the observable population DiD were incorrectly interpreted as the ATT or the ATE rather than MATTS, then one would wrongly reject the hypothesis that a unit tax reduces firm output, and hence wrongly reject the profit maximizing assumption. I also apply the results of Section 5 to find conditions under which MATTS is negative.

7.1.1 Firms

Consider a population of n firms where each firm i selects output $y_i \ge 0$ to maximize profit π^i . Firm i's cost $c_i(y_i) + \gamma_i \lambda_i y_i$ depends on its output and a unit tax $\lambda_i > 0$. Production cost $c_i(y_i)$ is convex, $c''_i(y_i) \ge 0$ for all $y_i \ge 0$ and all i. The subscript on λ_i allows the tax to be firm-specific. The parameter $\gamma_i > 0$ allows for the actual tax treatment to differ in magnitude, but not the sign, from the intended treatment.

7.1.2 Imperfect Competition

First consider a model with imperfect competition in which inverse demand for each firm is linear, $p_i(y) = a_i - \sum_{j \neq ij=1}^n b_{ij}y_j - \frac{1}{2}b_{ii}y_i$ with $b_{ii} > 0$ for all *i*. Profit is $\pi^i = p_i(y)y_i - c_i(y_i) - \gamma_i\lambda_iy_i$.

Assume a unique interior equilibrium exists. The first order condition for each firm i = 1, ..., n is

$$\frac{\partial \pi^i}{\partial y_i} = a_i - \sum_{j=1}^n b_{ij} y_j - c'_i(y_i) - \gamma_i \lambda_i = 0.$$

Equilibrium $\bar{y} = (\bar{y}_1, ..., \bar{y}_n)$ is a solution to this system of *n* equations. Define $f^i \equiv \frac{\partial \pi^i}{\partial y_i}$ to match the notation from Section 3. The negated Jacobian of the system is

$$-D_y f(\bar{y}; \bar{\lambda}) = \begin{bmatrix} b_{11} + c_1''(\bar{y}_1) & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & + c_2''(\bar{y}_2) \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} + c_n''(\bar{y}_n) \end{bmatrix}$$

Notice that the off-diagonal terms of the negated Jacobian can be interpreted as the slope coefficients of firm demand, or as the change in marginal profit because $\frac{\partial^2 \pi_i}{\partial y_i \partial y_j} = -b_{ij}$. The latter interpretation generalizes so I use it. The typical element in the vector of direct treatment effects $D_{\lambda} f(\bar{y}; \bar{\lambda})$ is $-\gamma_i < 0$.

A perverse example. Let n = 3. Set $c_i(y_i) \equiv 0$ and $\gamma_i = 1$ for i = 1, 2, 3. Suppose

$$p_1(y) = a - \frac{1}{6}y_1$$
, $p_2(y) = a - \frac{1}{2}y_2$, and $p_3(y) = a + \frac{5}{4}y_1 + \frac{5}{4}y_2 - y_3$.

for a positive and large enough. Firm 3's demand is complementary with firm 1 and 2's output, but firm 1 and firm 2's demand is independent of the others. These stark assumptions are designed to illustrate the mechanics of the example.

Equilibrium is the solution to the system of first order conditions:

$$\pi_1^1(y;\gamma_1,\lambda_1) = a - \frac{1}{3}y_1 - \lambda_1 = 0,$$

$$\pi_1^2(y;\gamma_2,\lambda_2) = a - y_2 - \lambda_2 = 0, \text{ and}$$

$$\pi_1^3(y;\gamma_3,\lambda_3) = a + \frac{5}{4}y_1 + \frac{5}{4}y_2 - 2y_3 - \lambda_3 = 0.$$
(7.1)

The equilibrium quantities are

$$\bar{y}_1 = 3a - 3\lambda_1, \ \bar{y}_2 = a - \lambda_2, \ \text{and} \ \bar{y}_3 = 3a - \frac{15}{8}\lambda_1 - \frac{5}{8}\lambda_2 - \frac{1}{2}\lambda_3.$$

If firms 1 and 2 are treated while firm 3 is untreated, then

$$\frac{d\bar{y}_1}{d\lambda} = -3, \ \frac{d\bar{y}_2}{d\lambda} = -1, \ \text{and} \ \frac{d\bar{y}_3}{d\lambda} = -\frac{5}{2}.$$

Each firm reduces output in equilibrium, yet

$$MATTS = \frac{-3-1}{2} - \left(-\frac{5}{2}\right) = \frac{1}{2} > 0$$

In practice, a researcher estimates the population DiD which in turn identifies MATTS under (AS1) and (AS2). But if the researcher were to interpret the population DiD as the ATT (or ATE), then the researcher would erroneously reject the hypothesis that the tax decreases output among the taxed, on average, and thus reject the profit maximizing hypothesis.

One may take issue in this example with the fact that in firm 3's demand function, the slope coefficients on firm 1 and 2's output (-5/4) is larger in magnitude than the slope coefficient on firm 3's output (1). There are two responses to this concern. First, this type of situation is ruled out in order to guarantee that MATTS is strictly negative. Second, I can extend this example to n firms where firms 1 to n - 1experience no spillover effects $(b_{ij} = 0 \text{ for } i = 1, ..., n - 1 \text{ and } j \neq i)$ and firm nexperiences anonymous spillover effects $(b_{nj} = \beta_n < 0 \text{ for all } j \neq n)$. Then, by Corollary 2, MATTS is strictly negative if and only if $b_{nn} < -(n-1)\beta = (n-1)|\beta|$. Thus, whenever n > 3 we can have $|\beta| < b_{nn}$ and MATTS strictly positive.

Another possible objection to this example is that, in the context of random assignment, the heterogeneity in firms' strategic interaction by realized treatment status would likely be detected in some balance test. However, each group assignment vector is equally likely under random assignment, and the results in this paper restrict the sign of MATTS for *any* assignment. Thus, if the randomization procedure is known to be sound, sign restrictions on MATTS can, and in fact *should*, be tested despite the outcome of any balance test.

Disciplining MATTS. I now apply the results of Section 5 to find conditions under which MATTS is negative. In each case below, Corollary 1 applies and other traditional comparative statics results are available—I refer the reader to the original Theorems for them. Recall that these results are valid for *any* treated subset of firms.

• Corollary 2. Suppose spillovers are anonymous-by-unit, $-b_{ij} = -\beta_i \ \forall i, j \neq i$. Then MATTS is strictly negative if, and only if, $\forall i$,

$$\begin{cases} b_{ii} + c_i''(\bar{y}_i) > -\beta_i & \text{if } -\beta_i > 0\\ b_{ii} + c_i''(\bar{y}_i) > -(n-1)\beta_i & \text{if } -\beta_i \le 0 \end{cases}$$

• Theorem 4. Suppose spillovers are positive, $-b_{ij} \ge 0 \ \forall i, j \ne i$. Then MATTS is strictly negative if

$$b_{ii} + c_i''(\bar{y}_i) > (n-1) \max_{j \neq i} \{-b_{ij}\} \quad \forall i.$$
 (7.2)

• Theorem 5 and Lemma 2. Suppose spillovers are strictly negative, $-b_{ij} < 0$ $\forall i, j \neq i$. Then MATTS is strictly negative if, $\forall i, j \neq i$,

$$0 < m \le \frac{b_{ij}}{(b_{ii} + c_i''(\bar{y}_i))} \le M < 1,$$

where $M^2 = sm + (1 - s)m^2$ with $s \leq \frac{1}{n-2}$. Alternatively, MATTS is strictly negative if m = M or n = 2.

• Theorem 6. MATTS is strictly negative if

$$b_{ii} + c_i''(\bar{y}_i) > (n-1)^2 \sum_{j \neq i} |-b_{ij}| \quad \forall i.$$
 (7.3)

The impact of increasing output on own marginal profit is $\frac{\partial^2 \pi^i}{\partial y_i^2} = b_{ii} + c''_i(\bar{y}_i)$. These restrictions thus limit the impact on a firm's marginal profit of increases in rivals' output relative to an increase in own output.

Because $\frac{\partial^2 \pi_i}{\partial y_i \partial y_j} = -b_{ij}$, it follows that competition is characterized by strategic complements if $-b_{ij} \ge 0$ for all $i, j \ne i$. In this case, Corollary 2 and Theorem 4 apply. If interactions are also anonymous-by-unit, the condition in Theorem 4 imposes a penalty of (n-1) if we fail to take into account this homogeneity in strategic effects. Observe that Theorem 6 also applies, but condition (7.3) is stronger than condition (7.2) because it fails to account for sign homogeneity.

Competition is characterized by strategic substitutes if $-b_{ij} \leq 0$ for all $i, j \neq i$. In this case, Corollary 2 and Theorems 5 and 6 apply. The textbook homogeneous goods Cournot oligopoly features $-b_{ij} = -\beta < 0$, with $b_{ii} = 2\beta$ and $c_i(y_i) = c(y_i)$. This satisfies the conditions of Theorem 5 (m = M). Also observe that if interactions are anonymous-by-unit, condition (7.3) imposes a penalty of $(n - 1)^2$ if we fail to account for this homogeneity.

Generalization. While this analysis assumed linear demand and convex costs, it generalizes easily. Specifically, the negated Jacobian is the matrix of cross-partials $-\frac{\partial^2 \pi_i}{\partial y_i \partial y_j}$. The results obtain if in each of the conditions we replace b_{ij} with $-\frac{\partial^2 \pi_i}{\partial y_i \partial y_j}$ for the off-diagonal terms $(i \neq j)$ and replace the diagonal terms, $b_{ii} + c''_i(\bar{y}_i)$, with $-\frac{\partial^2 \pi_i}{\partial y_i^2}$.

7.1.3 Perfect Competition

Now suppose firms are price-takers and produce a homogeneous good. Firms select output to maximize profit taking price as given,

$$\max_{y_i} py_i - c_i(y_i) - \gamma_i \lambda_i y_i$$

Assume $c_i(y_i)$ is strictly increasing, strictly convex, and that $c''_i(0) = 0$. The profitmaximizing quantities satisfy

$$p - c'_i(y_i) - \gamma_i \lambda_i = 0$$
 for $i = 1, ..., n$.

Let inverse market demand be $p = D(\sum_{i=1}^{n} y_i)$ for $D : \mathbb{R} \to \mathbb{R}$ a strictly decreasing and differentiable function. Recall $Y \equiv \sum_{i=1}^{n} y_i$ and substitute this into the last display to write

$$D(Y) - c'_i(y_i) - \gamma_i \lambda_i = 0 \text{ for } i = 1, ..., n.$$

Equilibrium outputs are a solution to this system. The elements on the main diagonal of the negated Jacobian are $-D'(\bar{Y}) + c''_i(\bar{y}_i)$ and all of the off-diagonal terms equal $-D'(\bar{Y})$. Thus, Theorem 5 applies if costs are convex, $c''_i(\bar{y}_i) > 0$ for all $i.^{14}$ Any tax increase on any nontrivial subset of firms implies that MATTS is strictly negative and total output decreases. Hence, market price increases whenever any subset of firms is taxed. Moreover, the output of every treated firm (i.e., those whose tax increases) decreases and the output of every untreated firm increases. In this

¹⁴If $c_i''(\bar{y}_i) \leq 0$ for some *i* then Theorem 5.1 can be applied if $-D'(\bar{Y}) + c_i''(\bar{y}_i) > 0$.

way, perfect competition is the ideal setting in which to test the profit maximization hypothesis via the comparative statics of taxation.

7.1.4 Discussion

While many models of firm interaction—such as perfect competition and the textbook Cournot oligopoly—guarantee that MATTS is negative, the perverse example demonstrates that MATTS can be positive in some settings. Thus, the Silberberg-Samuelson proposed test of the profit maximization hypothesis must also account for firm interaction if the researcher employs the DiD framework. If the estimate of MATTS is statistically significant and positive for *any* treatment group, then we can reject the joint assumptions concerning profit maximization and firm interaction.

Existing comparative statics results for oligopoly do not focus on MATTS. Moreover, these papers often impose versions of diagonal dominance, symmetry, aggregative interaction effects, strategic complements or strategic substitutes.¹⁵ These assumptions amount to different ways of limiting strategic heterogeneity. Although I impose differentiability at equilibrium, the framework can answer traditional comparative statics questions and is flexible enough to allow for these assumptions, as well as other types of strategic (and nonstrategic) interaction. Moreover, this unified approach clarifies how strategic heterogeneity influences comparative statics.

7.2 Symmetry, Anonymous-by-Unit Spillovers, and Contests

In this section I use contests to illustrate how symmetric equilibria give rise to anonymous-by-unit spillovers. There are *n* contestants who exert effort y_i to win a prize whose value is $V_i > 0$ to contestant *i*. The *contest success function (CSF)* is the probability that contestant *i* wins the prize, $p_i(y)$, as a function of the effort vector *y*. Letting $c_i(y_i)$ be the cost of effort, the expected payoff to contestant *i* is

$$u_i(y) = p_i(y)V_i - c_i(y_i).$$

 $^{^{15}}$ See, for example, Dixit (1986), Gama and Rietzke (2019), Acemoglu and Jensen (2013), Topkis (2011), L. C. Corchón (1994), Jinji (2014), Vives (1990), Vives (1999), and Milgrom and Roberts (1990).

Assume that the CSF is symmetric in y_{-i} . That is, for all i = 1, ..., n,

$$p_i(y_i; y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n) = p_i(y_i; y_{\pi(1)}, \dots, y_{\pi(i-1)}, y_{\pi(i+1)}, \dots, y_{\pi(n)})$$

for any permutation $\pi: N \to N$ of efforts other than *i*. The logit form (e.g., Dixit, 1987),

$$p_i(y) = \frac{h(y_i)}{r_i + \sum_{j=1}^n h(y_j)},$$
(7.4)

where h is an increasing function and $r_i \ge 0$ is a discount rate, is symmetric in this sense. Tullock (1980) studies the case where $h(y_i) = y_i^{\alpha}$ for $\alpha > 0$ and $r_i = 0 \forall i$.¹⁶

While the results of the paper can be applied to asymmetric equilibria, I focus on symmetric, interior equilibria to highlight a particularly powerful result. In this case, the FOCs are

$$f^{i}(y) \equiv \frac{\partial u_{i}(y)}{\partial y} = \frac{\partial p_{i}(y)}{\partial y_{i}} V_{i} - c'_{i}(y_{i}) = 0 \text{ for } i = 1, ..., n,$$

and equilibrium \bar{y} is a solution to this system with the property that $\bar{y}_i = y^*$ for all *i*. Then $-D_y f^i(\bar{y})$ has typical diagonal and off-diagonal elements

$$-f_i^i(\bar{y}) = -\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i^2} V_i + c_i''(\bar{y}_i) \text{ and } -f_j^i(\bar{y}) = -\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j} V_i,$$

respectively. Assume $f_i^i(\bar{y}) < 0$, which is satisfied if $p_i(y)$ is concave in own effort y_i at \bar{y} and $c_i(y_i)$ is convex at \bar{y}_i . Observe that the symmetry of $p_i(y)$ in y_{-i} is preserved in its own partial derivative, $\frac{\partial p_i(y)}{\partial y_i}$.¹⁷ Then the cross-partials evaluated at a symmetric equilibrium have the property that $f_j^i(\bar{y}) = f_k^i(\bar{y})$ for all i and all $j \neq k \neq i$.¹⁸ Hence, spillovers are anonymous-by-unit at equilibrium and we can apply the results in Section 5.1.

To this end, suppose that some subset of contestants is treated and that the direct treatment effect is positive for any treated contestant. Examples of such treatments

¹⁶See Beviá and L. Corchón, 2024 for a discussion of other CSFs.

¹⁷To see this, note that symmetry of $p_i(y)$ in y_{-i} implies $p_i(y_i + h, y_{-i}) = p_i(y_i + h, y_{\pi(-i)})$ for any $h \in \mathbb{R}$. Thus, $\lim_{h\to 0} \frac{p_i(y_i+h,y_{-i})-p_i(y_i,y_{-i})}{h} = \lim_{h\to 0} \frac{p_i(y_i+h,y_{\pi(-i)})-p_i(y_i,y_{\pi(-i)})}{h}$. ¹⁸To see this, let y_{-ijk} denote efforts with indexes other than i, j, or k. Then symmetry in y_{-i} and

¹⁸To see this, let y_{-ijk} denote efforts with indexes other than i, j, or k. Then symmetry in y_{-i} and the fact that $\bar{y}_j = \bar{y}_k = y^*$ imply $\frac{\partial p_i}{\partial y_i} (\bar{y}_i; \bar{y}_j + h, \bar{y}_k, \bar{y}_{-ijk}) = \frac{\partial p_i}{\partial y_i} (\bar{y}_i; \bar{y}_j, \bar{y}_k + h, \bar{y}_{-ijk})$ for any $h \in \mathbb{R}$. Thus, $\lim_{h\to 0} \frac{\frac{\partial p_i}{\partial y_i} (\bar{y}_i; \bar{y}_j + h, \bar{y}_k, \bar{y}_{-ijk}) - \frac{\partial p_i}{\partial y_i} (\bar{y}_i; \bar{y}_j, \bar{y}_k, \bar{y}_{-ijk})}{h} = \lim_{h\to 0} \frac{\frac{\partial p_i}{\partial y_i} (\bar{y}_i; \bar{y}_j, \bar{y}_k + h, \bar{y}_{-ijk}) - \frac{\partial p_i}{\partial y_i} (\bar{y}_i; \bar{y}_j, \bar{y}_k, \bar{y}_{-ijk})}{h}$.

include an increase in the value of the prize V_i , an increase in the probability of winning $p_i(y)$, or a decrease in the effort cost. Then by Corollary 2, MATTS is positive if, and only if, $-D_y f^i(y)$ is a *B*-matrix, or

$$\begin{cases} \left(\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i^2} - \frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j}\right) V_i < c''_i(\bar{y}_i) & \text{if } \frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j} V_i \le 0 \text{ and} \\ \left(\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i^2} + (n-1)\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j}\right) V_i < c''_i(\bar{y}_i) & \text{if } \frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j} V_i > 0. \end{cases}$$

Suppose costs are convex at \bar{y} , $c_i''(\bar{y}_i) \geq 0$. Then if the marginal win probability decreases with rivals' effort, this assumption is satisfied if $\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i^2} < \frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j}$, that is, if increasing own effort lowers the marginal win probability by more than it decreases when *any* rival increases effort. On the other hand, if the marginal win probability increases with rival effort, this assumption is satisfied if $\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i^2} < -(n-1)\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j}$, that is, if an increase in own effort lowers the marginal win probability by more than it increases when *every* rival increases effort.

If these conditions are met then Corollary 1 and Theorem 3 apply as well. We conclude that if only one contestant receives treatment, then her effort increases by a magnitude greater than by which any other contestant's effort changes. If one or more contestant is treated, then both the average effort among the treated and the total effort increases. If, in addition, $\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j} < 0$ for all *i*, then the average effort among the untreated decreases; while if $\frac{\partial^2 p_i(\bar{y})}{\partial \bar{y}_i \partial \bar{y}_j} > 0$ for all *i*, then the average effort among the untreated increases. In the latter case, the average effort among the treated increases by more than the average effort among the untreated because MATTS is positive.

7.2.1 Discussion

Nti (1997), Gama and Rietzke (2019), and Escobar (2025) study comparative statics in contests in the case where every player is treated simultaneously. The predictions from this type of analysis are not testable using reduced form methods because treated and untreated groups cannot be compared. Acemoglu and Jensen (2013) allow for variations of (7.4) which preserve its aggregative structure. In contrast, the analysis in this section allows for any CSF which is symmetric in y_{-i} , and other results in this paper can be applied to the nonsymmetric case. None of these papers directly study the comparative statics of MATTS or average outcomes.

The symmetric case is of fundamental interest in the contest literature because it is so widely studied. Whether this model is empirically relevant remains an open question, as there are several difficulties with estimation in no small part due to spillovers (Jia et al., 2013). However, as long as "effort" can be measured convincingly, this paper shows that the symmetric model is rejectable using a DiD framework by testing a sign restriction on MATTS.

I also note that the analysis for symmetric contests at symmetric equilibria translates directly to any symmetric game at symmetric equilibria where the IFT can be applied.¹⁹ For then the first-order conditions are symmetric in y_{-i} and spillovers are anonymous-by-unit at equilibrium. In fact, fully symmetric payoffs are not necessary, but only that payoffs are symmetric in y_{-i} . Moreover, this insight can be applied to any model in which equation $f^i(y; \lambda)$ is symmetric in y_{-i} , not just games. For example, the model of perfect competition in Section 7.1.3 is symmetric in this sense.

8 Conclusion

This paper expands our ability to test economic theory using reduced form estimation. It takes another step towards fulfilling Samuelson's (1947) vision that comparative statics analysis should be able to deliver "fruitful" or "meaningful" theorems which can be empirically rejected.²⁰ The key findings are that the sample DiD is an unbiased estimator of MATTS even when spillovers are unrestricted and, somewhat surprisingly due to the *a priori* complexity of the problem, the sign of MATTS can be restricted for *any* treatment group. Moreover, in many cases the conditions under which MATTS can be signed are the same or only slightly stronger than the conditions under which traditional comparative statics can be signed for general n.

In addition to its function in empirical hypothesis testing, MATTS is also useful in understanding the mechanisms and implications of a model. The analysis yields new insights to old questions while also posing and answering new questions. These insights are facilitated by the observation that B-matrices play an important role in signing comparative statics. Developing this connection requires us to derive new results for the class of B-matrices, especially with respect to conditions on the elements of a matrix under which its inverse is a B-matrix by columns.

¹⁹See p. 115 in Moulin (1986) for conditions under which a symmetric game has a symmetric equilibrium. Note that the smoothness assumptions associated with the IFT need only be applied at the equilibrium, so this allows for payoff functions which are not globally differentiable.

 $^{^{20}}$ I opt for the term *rejected* rather than *refuted* because Type I errors are not ruled out by the methods outlined in this paper.

A Appendix

Proof of Theorem 1. (1) For a given $g \in \Gamma$,

$$ATTS = \frac{1}{n_t} \sum_{r:g_r=1} \left(\sum_{s:g_s=1} \delta_{rs} f^s_\lambda \right) = \frac{1}{n_t} \sum_{s:g_s=1} \left(\sum_{r:g_r=1} \delta_{rs} \right) f^s_\lambda \ge 0$$

whenever $f_{\lambda}^{s} \geq 0$ for all $s: g_{s} = 1$ iff $\sum_{r:g_{r}=1} \delta_{rs} \geq 0$ for all $s: g_{s} = 1$. Since the result is for any $g \in \Gamma$, it follows that $ATTS \geq 0$ whenever $D_{\lambda}f \geq 0$ iff $\sum_{r:g_{r}=1} \delta_{rs} \geq 0$ for all $s: g_{s} = 1$ and all $g \in \Gamma$.

In addition, it is easy to see that, for a given $g \in \Gamma$, ATTS > 0 whenever $f_{\lambda}^{s} > 0$ for all $s : g_{s} = 1$ iff $\sum_{r:g_{r}=1} \delta_{rs} \geq 0$ for all $s : g_{s} = 1$ with strict inequality for at least one $s : g_{s} = 1$. The result follows since this must be true for any $g \in \Gamma$.

(2) The proof is similar to that of part (1) when $n_t < n$. If $n_t = n$, then ASU = 0 by assumption.

(3) For a given $g \in \Gamma$ such that $n_t < n$,

$$MATTS = \frac{1}{n_t} \sum_{s:g_s=1} \left(\sum_{r:g_r=1} \delta_{rs} \right) f_{\lambda}^s - \frac{1}{n_u} \sum_{s:g_s=1} \left(\sum_{r:g_r=0} \delta_{rs} \right) f_{\lambda}^s$$
$$= \sum_{s:g_s=1} \left(\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} - \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs} \right) f_{\lambda}^s \ge 0$$

whenever $f_{\lambda}^s \ge 0$ for $s: g_s = 1$ iff $\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \ge \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}$ for all $s: g_s = 1$. If $n_t = n$ so that g = (1, 1, ..., 1), $MATTS = \frac{1}{n} \sum_{s=1}^n (\sum_{r=1}^n \delta_{rs}) f_{\lambda}^s \ge 0$ iff $\sum_{r=1}^n \delta_{rs} \ge 0$ for s = 1, ..., n.

Since the result is for any $g \in \Gamma$, it follows that $MATTS \ge 0$ whenever $D_{\lambda}f \ge 0$ iff $\sum_{r:g_r=1} \delta_{rs} \ge 0$ for all s = 1, ..., n. A similar argument proves the result which characterizes MATTS > 0.

Proof of Theorem 2. To prove this result, I will show that

$$\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \ge \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs} \text{ for all } s: g_s = 1, \text{ any } g \in \Gamma \text{ with } 1 \le n_t < n; \text{ and } (A.1)$$
$$\frac{1}{n} \sum_{r=1}^n \delta_{rs} \ge 0 \text{ for } s = 1, \dots, n$$
(A.2)

is equivalent to the definition of a B_0 -matrix by columns:

$$\frac{1}{n}\sum_{r=1}^{n}\delta_{rs} \ge c_{s}^{+} \text{ for } s = 1, ..., n.$$
(A.3)

The result for *B*-matrices is obtained by using strict rather than weak inequalities in the " \Leftarrow " direction.

 (\Rightarrow) Fix $g \in \Gamma$. If $n_t = n - 1$, then (A.1) implies, for each s = 1, ..., n,

$$\frac{1}{n-1}\sum_{r:g_r=1}\delta_{rs} \ge \delta_{ks} \text{ for } k: g_k = 0.$$

Because this is true for any assignment $g \in \Gamma$, then for each s, k = 1, ..., n and $k \neq s$,

$$\frac{1}{n-1}\sum_{r\neq k,r=1}^n \delta_{rs} \ge \delta_{ks} \iff \sum_{r\neq k,r=1}^n \delta_{rs} \ge (n-1)\delta_{ks} \iff \sum_{r=1}^n \delta_{rs} \ge n\delta_{ks}.$$

Combining these n-1 inequalities with (A.2) results in (A.3).

(\Leftarrow) (A.3) directly implies (A.2). To show that (A.1) is also implied, let $1 \le n_t < n$ and fix $g \in \Gamma$. Then for s, k = 1, ..., n and $k \ne s$,

$$\sum_{r=1}^{n} \delta_{rs} \ge n \delta_{ks} \iff \sum_{r:g_r=1} \delta_{rs} \ge n \delta_{ks} - \sum_{r:g_r=0} \delta_{rs}.$$

Now fix a column s where $s : g_s = 1$ and sum over the $n_u > 0$ inequalities with k such that $g_k = 0$:

$$n_u \sum_{r:g_r=1} \delta_{rs} \ge n \sum_{k:g_k=0} \delta_{ks} - n_u \sum_{r:g_r=0} \delta_{rs}$$
$$n_u \sum_{r:g_r=1} \delta_{rs} \ge n_t \sum_{r:g_r=0} \delta_{rs}$$
$$\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \ge \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}.$$

Because this inequality holds for any $s : g_s = 1$ and any $g \in \Gamma$, it must hold for s = 1, ..., n, as desired.

Proof of Lemma 1. I establish a few facts to facilitate the proof. From $-D_y f[-D_y f]^{-1} = I$ it follows that, for $i \neq j$ and j = 1, ..., n, $\sum_{m=1}^n -f_m^i \delta_{mj} = -\alpha_i \delta_{ij} - \beta_i \sum_{m \neq i} \delta_{mj} = 0$. The last equality implies

$$\delta_{ij} = -\frac{\beta_i}{\alpha_i} \sum_{m \neq i} \delta_{mj}.$$
(A.4)

Next, it follows from (3.1) that $D(\bar{y})$ equals the j^{th} column of $-[D_y f]^{-1}$ when g has all components equal to zero, except for a 1 in the j^{th} position, and $D_{\lambda}f$ is a vector of ones. In this case, the i^{th} element of $D(\bar{y})$ is $\frac{dy_i}{d\lambda} = \delta_{ij}$, so by (5.4), the sum of each column j = 1, ..., n of $-[D_y f]^{-1}$ is

$$\sum_{m=1}^{n} \delta_{mj} = -\frac{1}{\Gamma} \frac{1}{\alpha_j - \beta_j}.$$
(A.5)

Next, for $i \neq j$ and j = 1, ..., n, we have from (A.4) that

$$\sum_{m \neq i} \delta_{mj} = \sum_{m=1}^{n} \delta_{mj} - \delta_{ij} = \sum_{m=1}^{n} \delta_{mj} + \frac{\beta_i}{\alpha_i} \sum_{m \neq i} \delta_{mj}.$$

This implies for $i \neq j$ and j = 1, ..., n that

$$\sum_{m \neq i} \delta_{mj} = \frac{\alpha_i}{\alpha_i - \beta_i} \sum_{m=1}^n \delta_{mj}.$$
 (A.6)

Finally, note that $\sum_{m=1}^{n} \delta_{mj} > n \delta_{ij}$ for $i \neq j$ iff $\sum_{m \neq i} \delta_{mj} > (n-1)\delta_{ij}$, which by (A.4) is equivalent to

$$\sum_{m \neq i} \delta_{mj} > -(n-1) \frac{\beta_i}{\alpha_i} \sum_{m \neq i} \delta_{mj}.$$
(A.7)

(⇒) Suppose $-D_{\lambda}f$ is a *B*-matrix. To show that $[-D_{\lambda}f]^{-1}$ is a *B*-matrix by columns, we must show $\sum_{m=1}^{n} \delta_{mj} > n \max\{0, \delta_{ij} | i \neq j\}$ for all *j*. In the main text it was shown in the paragraph after (5.4) that $\Gamma > 0$. We also have, for all *j*, $\alpha_j < 0$ and $\alpha_j - \beta_j < 0$ because $-D_{\lambda}f$ is a *B*-matrix. Thus, $\sum_{m=1}^{n} \delta_{mj} > 0$ by (A.5).

These facts and (A.6) also imply, for $i \neq j$ and j = 1, ..., n, that $\sum_{m\neq i} \delta_{mj} = \frac{\alpha_i}{\alpha_i - \beta_i} \sum_{m=1}^n \delta_{mj} > 0$. Then inequality (A.7) reduces to $\alpha_i < -(n-1)\beta_i$, which is

implied because A is a B-matrix. Thus, $\sum_{m=1}^{n} \delta_{mj} > n \delta_{ij}$ for all $i \neq j$ and j = 1, ..., n. Hence, $[-D_{\lambda}f]^{-1}$ is a B-matrix by columns.

(\Leftarrow) Now suppose $[-D_y f]^{-1}$ is a *B*-matrix by columns. We wish to show that

$$\begin{cases} \alpha_i < \beta_i & \text{if } \beta_i \le 0\\ \alpha_i < -(n-1)\beta_i & \text{if } \beta_i > 0. \end{cases}$$

To this end, note that (A.5) implies implies $\operatorname{sgn}\left(-\frac{1}{\Gamma}\right) = \operatorname{sgn}\left(\frac{1}{\alpha_j - \beta_j}\right) \forall j$. It follows that either $\alpha_j - \beta_j > 0 \; \forall j$ or $\alpha_j - \beta_j < 0 \; \forall j$.

Suppose $\alpha_j - \beta_j < 0 \ \forall j$. Because $[-D_y f]^{-1}$ is a *B*-matrix by columns, for j = 1, ..., n,

$$\sum_{m=1}^{n} \delta_{mj} > n \delta_{ij} \text{ for all } i \neq j.$$

Subtract δ_{ij} from both sides and then substitute (A.4) and (A.6) for δ_{ij} and $\sum_{m \neq i} \delta_{mj}$, respectively, to get, for j = 1, ..., n and all $i \neq j$

$$\sum_{\substack{m \neq i}} \delta_{mj} > (n-1)\delta_{ij}$$
$$\frac{\alpha_i}{\alpha_i - \beta_i} \sum_{m=1}^n \delta_{mj} > -(n-1)\frac{\beta_i}{\alpha_i - \beta_i} \sum_{m=1}^n \delta_{mj}$$

Divide both sides by $\sum_{m=1}^{n} \delta_{mj}/(\alpha_i - \beta_i)$ to get, for j = 1, ..., n and all $i \neq j$, $\alpha_i < -(n-1)\beta_i$. Hence, $-D_y f$ is a *B*-matrix.

Now suppose $\alpha_j - \beta_j > 0 \ \forall j$. The diagonal terms of the inverse of a *B*-matrix are positive (Christensen, 2019), so we must have $\alpha_j < 0$ since $\alpha_j \neq 0$ by assumption. It follows that $\beta_j < 0$ and (A.6) implies $\sum_{m \neq i} \delta_{mj} < 0$. Thus, (A.7) is equivalent to $\alpha_i > -(n-1)\beta_i > 0$ which contradicts the fact that $\alpha_i < 0$. Thus, $\alpha_j - \beta_j < 0 \ \forall j$.

Proof of Theorem 4. (1) By equation (3.1), $-D_y f D(\bar{y}) = G D_\lambda f$. Because $-D_y f$ is a Z-matrix and $G D_\lambda f \geq 0$, it follows that $D(\bar{y}) \geq 0$ if and only if $-D_y f$ is an *M*-matrix (Plemmons, 1977). This implies that ATTS and ASU are positive.

(2) By Theorem 2 and Corollary 1 we need to show that the inverse of $-D_y f$ is a *B*-matrix by columns, or $\sum_j \delta_{ij} > nc_j^+$ for all j = 1, ..., n.

As noted in the text following the Theorem, the conditions imply that $-D_y f$ is an *M*-matrix, so the elements of its inverse are all positive, $\delta_{ij} \geq 0 \ \forall i, j$. Then notice that $-D_y f[-D_y f]^{-1} = I$ implies that for all j = 1, ..., n and any $i \neq j$,

$$\begin{split} -f_i^i \delta_{ij} &= \sum_{m \neq i} f_m^i \delta_{mj} \leq \max_{m \neq i} \{f_m^i\} \sum_{m \neq i} \delta_{mj} \\ \Rightarrow (n-1) \delta_{ij} \leq \frac{n-1}{-f_i^i} \max_{m \neq i} \{f_m^i\} \sum_{m \neq i} \delta_{mj}. \end{split}$$

Because $-f_i^i > (n-1) \max_{m \neq i} \{f_m^i\}$ we have $\frac{n-1}{-f_i^i} \max_{m \neq i} \{f_m^i\} < 1$. Thus,

$$0 \le (n-1)\delta_{ij} < \sum_{m \ne i} \delta_{mj} \Longrightarrow 0 \le n\delta_{ij} < \sum_{m=1}^n \delta_{mj},$$

as desired.

Proof of Lemma 2. Let H be the diagonal matrix with diagonal elements $-1/f_i^i$. Then $-HD_yf(\bar{y};\bar{\lambda})$ is a strictly positive matrix with a unit diagonal. It follows from Willoughby (1977) that $-[HD_yf]^{-1} = -D_yf^{-1}H^{-1}$ is a SDD (by rows and columns) M-matrix. H^{-1} is a diagonal matrix with diagonal elements $-f_i^i > 0$, so $-D_yf^{-1}$ has the same sign pattern as $-D_yf^{-1}H^{-1}$. Then because $-D_yf \ge 0$, $-D_yf^{-1}$ is an M-matrix.

The fact that $-D_y f^{-1} H^{-1}$ is SDD by columns implies that for j = 1, ..., n,

$$\left|\delta_{jj}\right|\left(-f_{j}^{j}\right) = \left|\delta_{jj}f_{j}^{j}\right| > \sum_{i\neq j}\left|\delta_{ij}f_{j}^{j}\right| = \left(-f_{j}^{j}\right)\sum_{i\neq j}\left|\delta_{ij}\right|.$$

Hence, $|\delta_{jj}| > \sum_{i\neq j}^{n} |\delta_{ij}|$, which proves that $[-D_y f]^{-1}$ is SDD by columns. A similar argument shows that $[-D_y f]^{-1}$ is SDD by rows.

Proof of Theorem 5.

Lemma 2 implies $[-D_y f]^{-1}$ is SDD by columns. By the discussion in the main text immediately following Theorem 2, this implies $[-D_y f]^{-1}$ is a *B*-matrix by columns. Then by Theorem 2, MATTS> 0 for any assignment $g \in \Gamma$ whenever $D_{\lambda} f > 0$. The results of Corollary 1 therefore apply.

The fact that $-D_y f^{-1}$ is an *M*-matrix means that its off-diagonal terms are negative. Thus, Theorem 1 implies $ASU \leq 0$ for any assignment $g \in \Gamma$ whenever $D_\lambda f > 0$. Finally, $\frac{d\bar{y}_i}{d\lambda} = \sum_j \delta_{ij} f_\lambda^j g_j$ for all *i*. It follows from $\delta_{ij} \leq 0 \ \forall i \neq j$ that $\frac{d\bar{y}_i}{d\lambda} \leq 0 \ \forall i$ such that $g_i = 0$. And it follows from $\delta_{ii} > 0$ and the fact that $D_y f^{-1}$ is *SDD* by rows that $\frac{d\bar{y}_i}{d\lambda} > 0 \ \forall i$ such that $g_i = 1$ when $f_{\lambda}^j = \bar{f}_{\lambda} > 0$ for all j.

Proof of Theorem 6. Given (5.8), the fact that $-[D_y f]^{-1}$ is a *SDD* matrix by columns is immediate. To see that it is also a *B*-matrix by columns, observe that

$$\sum_{i} \delta_{ij} = \delta_{jj} + \sum_{i \neq j} \delta_{ij} > (n-1) \sum_{i \neq j} |\delta_{ij}| + \sum_{i \neq j} \delta_{ij} = \sum_{i \neq j} ((n-1) |\delta_{ij}| + \delta_{ij}) \ge 0,$$

because each term in the last sum is positive. Moreover, for every $k \neq j$,

$$\delta_{jj} + \sum_{i \neq j} \delta_{ij} - n \delta_{kj} > \sum_{i \neq j} ((n-1) |\delta_{ij}| + \delta_{ij}) - n \delta_{kj}$$

= $(n-1) |\delta_{kj}| - (n-1) \delta_{kj} + \sum_{i \neq j,k} ((n-1) |\delta_{ij}| + \delta_{ij}) \ge 0.$

It follows that, for j = 1, ..., n, $\sum_i \delta_{ij} > nc_j^+$, as desired. To prove that inequalities (5.7) imply that $-D_y f$ is a *B*-matrix, use the same approach but replace (n-1) with $(n-1)^2$ in the first inequality of each step.

I now show that inequalities (5.7) imply inequalities (5.8). Because $-D_y f$ is a *B*-matrix, it is invertible, has a strictly positive diagonal, and $[-D_y f]^{-1}$ has a strictly positive diagonal, or $\delta_{jj} > 0 \forall j$. Because it is SDD by rows, we know from Corollary 1 in Norris et al. (2023) that

$$\left|\delta_{ij}\right| < d^* \left|\delta_{jj}\right| = d^* \delta_{jj}$$

for all $j, i \neq j$, where $d^* = \max_i \left\{ \frac{\sum_{j\neq i} |-f_j^i|}{-f_i^i} \right\}$.²¹ If $d^* = 0$, then $-D_y f$ is a diagonal matrix and the result is trivial. Assume $d^* > 0$. Then for any j, take the sum over $i \neq j$ to get

$$\sum_{i \neq j} |\delta_{ij}| < (n-1)d^*\delta_{jj} \implies \frac{1}{(n-1)d^*} \sum_{i \neq j} |\delta_{ij}| < \delta_{jj}.$$

²¹In fact, Norris et al. (2023) assume the Jacobian is SDD by columns and prove the analogous implication holds for off-diagonal elements of a given row of the Jacobian inverse. By transposing the Jacobian so that it is SDD by rows, their result applies to the transposed Jacobian inverse. That is, it applies to the off-diagonal elements of a given column of the Jacobian inverse, as it is stated here. Their result improves on the bound from Ostrowski (1952) who showed that $\delta_{jj} > |\delta_{ij}|$ for all $j, i \neq j$.

Condition (5.7) implies $\frac{1}{(n-1)d^*} > n-1$. This completes the proof.

Proof of Lemma 3. We first show that randomization implies parallel trends. To this end, define $Y_{it}^{\Delta}(0) = Y_{i2}(0) - Y_{i1}(0)$. There are $\binom{n}{n_t}$ ways to draw n_t units. Let $D_j(n_t)$ be the set of n_t indexes in draw j. The expected value of $Y_{it}^{\Delta}(0)$ in draw j is $\frac{1}{n_t} \sum_{i \in D_j} Y_{it}^{\Delta}(0)$. It follows that

$$\mathbb{E}[Y_{it}^{\Delta}(0)|g_i=1] = \frac{1}{\binom{n}{n_t}} \sum_j \left(\frac{1}{n_t} \sum_{i \in D_j(n_t)} Y_{it}^{\Delta}(0)\right) = \frac{1}{\binom{n}{n_t}} \frac{1}{n_t} \sum_j \left(\sum_{i \in D_j(n_t)} Y_{it}^{\Delta}(0)\right)$$

Across the set of all draws of size n_t , a given index i appears $\binom{n-1}{n_t-1}$ times. To count this, note that if i appears in a draw, then there (n-1) indexes left to select for the remaining $(n_t - 1)$ indexes. This is so for each $i \in \{1, ..., n\}$. Continuing the chain of equalities from above we thus have

$$\mathbb{E}[Y_{it}^{\Delta}(0)|g_{i}=1] = \frac{1}{\binom{n}{n_{t}}} \frac{1}{n_{t}} \binom{n-1}{n_{t}-1} \sum_{i=1}^{n} Y_{it}^{\Delta}(0)$$
$$= \frac{n_{t}!(n-n_{t})!}{n!} \frac{1}{n_{t}} \frac{(n-1)!}{(n_{t}-1)!(n-n_{t})!} \sum_{i=1}^{n} Y_{it}^{\Delta}(0)$$
$$= \frac{1}{n} \sum_{i=1}^{n} Y_{it}^{\Delta}(0)$$
$$= \mathbb{E}[Y_{it}^{\Delta}(0)].$$

Because this result holds for an arbitrary but fixed number of units, it follows that $\mathbb{E}[Y_{it}^{\Delta}(0)|g_i=0] = \mathbb{E}[Y_{it}^{\Delta}(0)]$ as well. Thus, $\mathbb{E}[Y_{it}^{\Delta}(0)|g_i=1] = \mathbb{E}[Y_{it}^{\Delta}(0)|g_i=0]$, as desired.

To prove $\mathbb{E}[Y_{i1}(0)|g_i=1] = \mathbb{E}[Y_{i1}(0)|g_i=0]$ under randomization, swap $Y_{it}^{\Delta}(0)$ for $Y_{i1}(0)$ in the previous argument.

Proof of Theorem 7. The no-anticipatory-effects assumption (AS1) implies²²

$$\mathbb{E}[Y_{i1}(0)|g_i=1] = \mathbb{E}[Y_{i1}(g)|g_i=1] \text{ and } \mathbb{E}[Y_{i1}(0)|g_i=0] = \mathbb{E}[Y_{i1}(g)|g_i=0].$$

Substitute this into assumption (AS2) and rearrange to show that, in the absence of treatment, the difference in average outcomes between the treated and untreated in period 2 is identified by the observed difference in period 1:

$$\mathbb{E}[Y_{i2}(0)|g_i=0] - \mathbb{E}[Y_{i2}(0)|g_i=1] = \mathbb{E}[Y_{i1}(g)|g_i=0] - \mathbb{E}[Y_{i1}(g)|g_i=1].$$

Now re-arrange the right hand side of (6.1), substitute in the last result, and re-arrange a final time to prove part 1:

$$MATTS(g) = (\mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0]) + (\mathbb{E}[Y_{i2}(0)|g_i = 0] - \mathbb{E}[Y_{i2}(0)|g_i = 1])$$

$$= (\mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0]) + (\mathbb{E}[Y_{i1}(g)|g_i = 0] - \mathbb{E}[Y_{i1}(g)|g_i = 1])$$

$$= \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 0]$$

$$= DiD(g).$$

To prove part 2, note that, by the no-anticipatory-effects assumption, (AS3) is equivalent to $\mathbb{E}[Y_{i1}(g)|g_i = 1] = \mathbb{E}[Y_{i1}(g)|g_i = 0]$. Thus, under parallel trends, the DiM identifies MATTS:

$$\begin{aligned} MATTS(g) &= DiD(g) \\ &= \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 0] \\ &= \mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0] - (\mathbb{E}[Y_{i1}(g)|g_i = 1] - \mathbb{E}[Y_{i1}(g)|g_i = 0]) \\ &= \mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0] \\ &= DiM(g). \end{aligned}$$

Part 3 follows from Lemma 3 and part 2.

²²In fact, (AS1) can be relaxed to this form, but we keep (AS1) for expositional convenience.

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