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Comparative Statics for Difference-in-Differences

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Abstract

The stable unit treatment value assumption (SUTVA) in causal estimation rules out spillover effects, but spillover effects are the hallmark of many economic models. Testing model predictions with techniques that employ SUTVA are thus problematic. To address this issue, we first show that without the no interference component of SUTVA, the population difference-in-difference (DiD) identifies the difference in the average potential outcomes between the treated and untreated. We call this estimand the marginal average treatment effect among the treated with spillovers (MATTS). Then, in the context of a model whose equilibrium is characterized by a system of smooth equations, we provide comparative statics results which restrict the sign of MATTS. Specifically, we show that MATTS is positive for any nontrivial treatment group whenever treatment has a strictly positive direct effect if and only if the inverse of the negated Jacobian is a B_0-matrix by columns. We then provide several conditions on the Jacobian such that its negated inverse is a B-matrix by columns. Additional related results are presented. These predictions can be tested directly within the DiD framework even when the SUTVA is violated. Consequently, the results in this paper render economic models rejectable with reduced form DiD methods.

JEL Codes: C31, C33, C65, C72, D21, L21

Keywords: Comparative statics, difference-in-differences, SUTVA, spillovers, profit maximization hypothesis, refutability, B-matrix

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1 Introduction

The canonical difference-in-differences (DiD) framework invokes the stable unit treatment value assumption (SUTVA) which rules out spillover effects.[1] Spillovers are a key feature of many economic models, so this assumption constrains the set of testable comparative statics predictions using the DiD framework.

This paper ameliorates this issue in two steps. First, we reassess the causal interpretation of the observed population DiD when the no interference component of SUTVA is dropped. This process identifies a class of comparative statics predictions that are testable when spillovers are present. Second, we develop analytical results for this class of predictions for any model whose equilibrium can be expressed as a system of differentiable equations.

The DiD framework is used to test the effect of some policy or treatment on an outcome of interest. In the canonical approach, the sample is partitioned in the treated and untreated. Average outcomes by treatment status are calculated in period 1 before treatment and again in period 2 after treatment. The untreated group—sometimes called the control group—is used as a proxy for what would have happened to the units in the treatment group in period 2 if treatment had not occurred. The observed *population DiD* in this setting is the difference in the change in average outcomes among the treated and the change in average outcomes among the untreated. Under SUTVA, the population DiD identifies the *average treatment effect on the treated (ATT).*

Absent SUTVA, a policy treatment can have an impact on the untreated through spillover effects. For example, a unit tax applied to only some firms in an industry will increase market prices, and the higher prices will impact the strategic decisions of untaxed firms (the spillover effect on the untreated). When spillovers such as these are present, we show in Section 3 below that the observed population DiD identifies the difference between the average treatment effect on the treated with spillovers and the average spillover effect on the untreated, or the *marginal average treatment effect on the treated with spillovers (MATTS).*

If the population DiD is incorrectly interpreted as the ATT when spillovers are present, then it has a disturbing *sign reversal property*: the treatment effect can be positive for every unit while the population DiD is negative. One implication is that many traditional comparative statics predictions are untestable. A related implication is that canonical interpretation of the estimate of the population DiD may lead to a false rejection of a theory.

To illustrate these ideas, consider the first chapter of *The Structure of Economics* in which Silberberg (1978) demonstrates how comparatives statics analysis can be used to test

[1] Depending on the context, spillover effects may be called equilibrium effects, strategic effects, network effects, indirect effects, social interactions, peer effects, or something else.

the assumptions of a theory. In the leading example, the claim is that the profit maximizing assumption can be tested by evaluating a firm's response to a unit tax. A tax would cause the output of a profit maximizing firm to decrease. So, to simplify a bit, if instead we observe that output increases, then we can reject the profit maximizing assumption.

In practice, however, a test of this hypothesis needs observations of a firm at two points in time over which the tax varies. Other unobservable factors may change between these two points of observation, so the empiricist must control for these unobservables in some way. The DiD framework accomplishes this objective but requires a treated and untreated group, and thus a sample of more than one firm.

For simplicity, consider a 3-firm differentiated oligopoly in which the only variable that changes between two periods of observation is the unit tax. A formal model is developed in Section 8, but for now a summary discussion suffices. Suppose that the equilibrium effect of a unit tax on firms 1 and 2 (the treated group) is that their output decreases by 3 and 1 units, respectively, for an average decrease of 2 units. But due to strong demand complementarities (spillovers), firm 3 (the untreated group) decreases output by 2.5. Then the population DiD is $-2-(-2.5)=.5>0$. If this observed population DiD were incorrectly interpreted as the ATT rather than the MATTS, then one would wrongly reject the hypothesis that a unit tax reduces firm output, and hence wrongly reject the profit maximizing assumption.

Thus, the prediction that a tax on a single firm will reduce its output is fundamentally untestable because of spillovers. However, predictions on the sign of MATTS are testable, and these are the type of comparative static predictions this paper provides. In Section 8 we show that if *any* subset of firms is taxed, MATTS is negative if the market is perfectly competitive, or if the market is imperfect and the profit functions satisfy some additional conditions which constrain spillovers.

In addition to MATTS, we are intrinsically interested in whether a treatment has its hypothesized effect on the treated, so this paper also provides predictions on the sign of the impact of treatment on average outcomes among the treated, or the *average treatment effect on the treated with spillovers* (ATTS). In some cases we are also able to sign the impact on the average outcomes among the untreated, or the *average spillover effect on the untreated* (ASU), as well as some other comparative statics of interest. Note that MATTS is the difference between ATTS and ASU.

The setting is any model whose equilibrium is characterized by a system of equations. We interpret the index of each equation as a unit of observation. The comparative statics analysis is carried out by way of the implicit function theorem which involves the inverse of the Jacobian of the system. The Jacobian encodes spillover effects. The (i,j) and (j,i) off-diagonal terms of the Jacobian capture direct spillover effects between units i and j while the off-diagonal terms of its inverse capture equilibrium spillover effects.

In the discussion of the results that follows and unless otherwise specified, the results are valid for any nontrivial subset of units included in the treatment group, including all n units, as long as the direct treatment effect is positive. Intuitively, the direct treatment effect is positive for unit i if unit i's outcome would increase as a result of treatment if we were to shut down any spillover effects.

In Theorem 1 we show that ATTS is positive if and only if every diagonally-centered partial column sum of the Jacobian's negated inverse is positive, where *diagonally-centered* means that the sum includes the term on the main diagonal. Similarly, ASU is positive if and only if every non-diagonally-centered partial column sum of the Jacobian's negated inverse is positive. MATTS is positive if and only if ATTS is at least as large as ASU.

In Theorem 2 we show that MATTS is positive if and only if the negated Jacobian inverse is a B_0-matrix by columns. This is a central insight of the paper. First, it vastly simplifies the task of checking the predicted sign of MATTS. Second, in Corollary 1, we also show that the B_0-matrix by columns condition implies several additional comparative statics results: ATTS is positive, the sum of all outcomes increases, and, if only one unit is treated, then this unit's outcome increases by more than the magnitude of change in any other unit's outcome. Third, this result helps focus the analysis when seeking conditions on direct spillover effects (i.e., elements of the Jacobian) rather than equilibrium spillover effects (i.e., elements of the Jacobian inverse), such that the desired results obtain—analytically, we search for conditions under which a real matrix has an inverse that is a B_0-matrix by columns. We discuss these results next.

In Theorem 3 we assume direct spillovers that are anonymous-by-unit, meaning that for each unit i, a change in unit j's outcome has the same spillover impact on unit i's outcome as a change in unit k's outcome. This assumption provides remarkable structure if, in addition, the negated Jacobian is a B-matrix. Then MATTS and ATTS are strictly positive, the aggregate outcome increases with treatment, and the outcome of a singularly treated unit increases by more than the change in outcome of any untreated unit. The sign of ASU is ambiguous in general but positive (negative) if all off-diagonal terms are positive (negative). In addition, the B-matrix condition is necessary for MATTS to be strictly positive (Lemma 1 and Corollary 2).

In Theorem 4 we assume direct spillovers are positive, a case which arises in games with strategic complements. Here, an increase in one unit's outcome puts upward pressure on the outcome of all other units. We show that every unit's outcome increases whenever the negated Jacobian is an M-matrix, which can be interpreted as a type of stability condition. Since MATTS is the difference between ATTS and ASU, stronger conditions are required to ensure that MATTS is positive. These conditions also imply that a singularly treated unit's outcome increases by more than any other unit's outcome.

In Theorem 5 we assume direct spillovers are negative, a case which arises in games with strategic substitutes. In this case, an increase in one unit's outcome puts downward pressure on the outcome of all other units. Here we rely on a result from Willoughby (1977) under which the inverse of the negated Jacobian is a strictly diagonally dominant M-matrix. These conditions restrict the heterogeneity of normalized direct spillovers and imply that MATTS is strictly positive. Moreover, the outcome of any treated unit increases (and hence ATTS is strictly positive) while the outcome of any untreated unit weakly decreases (and hence ASU is negative). Also, a singularly treated unit's outcome increases by more than the change in any other unit's outcome.

Finally, in Theorem 6 we formalize the idea that intuitive comparative statics obtain when spillovers are small, no matter their sign. By intuitive comparative statics, we mean that MATTS and ATTS are strictly positive, the sum of outcomes increases, and a singularly treated unit's outcome increases by more than any other unit's outcome changes. Of the settings considered, this is the only one where the direct spillover effects acting within a unit can be both positive and negative.[2] However, this heterogeneity comes at a cost since the sufficient condition is stronger than the others.

The paper is organized as follows. In the next section we place our work in the literature. In Section 3 we show that the observed population DiD identifies MATTS when the no interference component of SUTVA is dropped. Section 4 presents the model used to derive comparative statics predictions. Section 5 provides some mathematical preliminaries. Section 6 gives conditions on the inverse of the Jacobian under which the desired comparative statics results hold, while Section 7 does this for the Jacobian. Section 8 provides an application of the results. Section 9 concludes.

2 Contributions to the Literature

This paper makes a conceptual and practical contribution on how to deal with spillovers and SUTVA. The problem with the SUTVA assumption is well-known (e.g., Manski, 1993; Rubin, 1986; Angrist, Imbens, and Rubin, 1996) but remains standard (Imbens and Rubin, 2015). Similar to Theorem 1 in Vazquez-Bare (2023) who studies randomized controlled trials, the approach in this project is to drop the no interference component of SUTVA and focus on what population estimand is identifiable. This approach is distinct from efforts to

[2]The anonymous-by-unit case allows direct spillovers to positive or negative depending on the unit, but they must be all positive or negative *within* a unit.

identify spillover and direct effects separately. To separately identify spillovers, researchers have relied on some combination of random assignment to spillovers, knowledge of the social network, and structural modeling in the sense that a specific model of spillovers is constructed and its parameters estimated.[3] Many of these papers use the linear-in-means framework. In contrast, the reduced form approach in this paper imposes very little structure for identification and allows us to say something meaningful when spillovers are present without having to separately identify them.

The second literature to which this paper contributes is the analysis of comparative statics at an equilibrium.[4] Due to the complexity of the problem that spillovers introduce, many papers in this literature examine the effect of a treatment applied to a single individual. An alternative approach is to restrict the sign of spillover effects to be either all positive or all negative. In this monotone case, predictions are typically made for the outcome of *each* individual. Neither approach is amenable to testing within the reduced form DiD framework since the treated and untreated groups typically include a large number of individuals for statistical consistency. This paper breaks new ground by posing and providing answers to the following testable question: When is the *average* impact of a treatment, or a shock, larger among treated units than among untreated units, regardless of the composition of the two groups?

In answering this new question, we also provide new insight into old questions, even while assuming differentiability. For example, we show that all the results concerning the impact of a parameter shock on equilibrium variables provided in Dixit (1986) for oligopoly are subsumed, and in fact much stronger results obtain under weaker conditions. The key insight of this paper that drives this and other generalizations is that several desirable comparative statics results obtain when the negated Jacobian inverse is a B-matrix by columns. Under various assumptions on spillovers, it turns out that this property of the inverse often obtains when the negated Jacobian is a type of B-matrix (by rows). This approach is in contrast to the prevailing one in the literature which relies on diagonal dominance. Finally, a well-cited and recent contribution to comparative statics analysis under the differentiability assumption is Acemoglu and Jensen (2013) who study aggregative games. For cases in which our context and questions context overlap, their conditions are weaker. However, our context does not

[3]A few examples include Sacerdote (2001), Duflo and Saez (2003), Bramoullé, Djebbari, and Fortin (2009), Carter, Laajaj, and Yang (2021), Blume, Brock, Durlauf, and Ioannides (2011), Blume, Brock, Durlauf, and Jayaraman (2015), Goldsmith-Pinkham and Imbens (2013), and Hirano and Hahn (2010).

[4]This literature is large. Generally speaking it can be classified into approaches which assume differentiability and those that do not. Some examples assuming differentiability include Dixit (1986), Nti (1997), Acemoglu and Jensen (2013), Christensen and Cornwell (2018), Christensen (2019), and Norris, Johnson, and Spitkovsky (2023), among others. Examples of lattice-theoretic or monotone methods include Topkis (2011), Milgrom and Roberts (1990), Amir (2005), and Vives (1990), among others.

require any aggregative structure to payoffs and we pose different questions.

Finally, the paper contains several new linear algebra results on B-matrices. Let $A = (a_{ij})$ be an $n \times n$ real matrix. Define

$$r_i^+ = \max\{0, a_{ij} | j \neq i\} \text{ and } c_j^+ = \max\{0, a_{ij} | i \neq j\}$$

as the largest nonnegative element in row i (column j). If all elements in the row (column) are strictly negative, set r_i^+ (c_i^+) equal to zero. Then A is a *B-matrix* if

$$\sum_{j=1}^{n} a_{ij} > n r_i^+. \tag{2.1}$$

for $i = 1, ..., n$. We say that A is a *B-matrix by columns* if

$$\sum_{i=1}^{n} a_{ij} > n c_j^+. \tag{2.2}$$

For each row of a B-matrix, the average of the entries is positive and greater than each of the off-diagonal entries. If inequalities (2.1) and (2.2) are weak, then A is a *B_0-matrix* and *B_0-matrix by columns*, respectively.

The class of B-matrices was introduced in the mathematical literature in Carnicer, Goodman, and Peña (1999) and Peña (2001), and was first applied to economics in Christensen (2019).[5] These linear algebra results are woven into the context of the comparative statics problem, so I highlighlight them here in notation that is friendlier for those interested in this mathematics.

Let us introduce some notation. Given $k, n \in \mathbb{N}$, $1 \leq k \leq n$, $Q_{k,n}$ will denote the set of all increasing sequences of natural numbers less than or equal to n. The element $\alpha_i \in Q_{k,n}$ contains the natural number i. Its complement $\alpha_i' \in Q_{n-k,n}$, which is the increasingly rearranged $\{1, 2, ..., n\} \backslash \alpha_s$, excludes i.

- (Theorem 2) For any $k = 1, ..., n-1$,

$$\frac{1}{k} \sum_{j \in \alpha_i} a_{ij} > (\geq) \frac{1}{n-k} \sum_{j \in \alpha_i'} a_{ij} \text{ for any } \alpha_i \in Q_{k,n} \text{ and}$$
$$\frac{1}{n} \sum_{j=1}^{n} a_{ij} > (\geq) 0 \text{ for } i = 1, ..., n$$

  if and only if A is a B-matrix (B_0-matrix).

[5] See also Hoffman (1965).

- (Lemma 1 part 2) Suppose $a_{ij} = \beta_i$ and $\alpha_{ii} \neq \beta_i$ for any $j \neq i$ and all $i = 1, ..., n$. Then A is a B-matrix if and only if A^{-1} is a B-matrix by columns.
- A^{-1} is a B-matrix by columns if any of the following conditions is satisfied:
  1. (Theorem 4) For all $i = 1, ..., n$, $a_{ij} \leq 0$ for any $j \neq i$ and
  $$a_{ii} > (n-1) \max_{j \neq i} \{-a_{ij}\}.$$
  Note that this condition implies that A is a B-matrix (and an M-matrix).
  2. (Theorem 5) Suppose $a_{ij} > 0$ for all i, j and the conditions given in Willoughby (1977) are satisfied. Willoughby (1977) showed that these conditions imply that A^{-1} is a strictly diagonally dominant (by rows and columns) M-matrix.
  3. (Theorem 6) For all $i = 1, ..., n$,
  $$a_{ii} > (n-1)^2 \sum_{j \neq i} |a_{ij}|.$$
  Note that this condition implies that A is a B-matrix.

3 DiD without SUTVA

We begin by reassessing the causal interpretation of the observed population DiD when the no interference component of SUTVA is dropped. Imbens and Rubin (2015) describe the no interference component of SUTVA as "the assumption that the treatment applied to one unit does not affect the outcome for other units." This is the component of SUTVA which rules out spillover effects.

Let us expand to allow for spillover effects the potential-outcomes-based presentation of DiD in Roth et al. (2023). Consider a balanced panel with two time periods, $t = 1, 2$. Units are indexed by $i = 1, ..., n$. Units in the treated group ($g_i = 1$) are treated only in period 2, whereas units in the untreated group ($g_i = 0$) are never treated.

Potential outcomes depend on the treatment status λ_{it} of each unit i in period t. λ_{it} is an indicator equal to 1 if unit i is treated in period t and 0 otherwise. To simplify notation, write $\lambda_i = 0$ if the unit is not treated in either period ($\lambda_{i1} = \lambda_{i2} = 0$) and $\lambda_i = 1$ if the unit is treated only in period 2 ($\lambda_{i1} = 0$ and $\lambda_{i2} = 1$).[6] In the observed data, $\lambda_i = 1$ only if unit i is in the treated group, $\lambda_i = g_i$; similarly, $\lambda_{it} = 1$ only if i is in the treated group and $t = 2$, $\lambda_{it} = (t-1)g_i$. However, we allow for $\lambda_i \neq g_i$ and $\lambda_{it} \neq (t-1)g_i$ to consider counterfactuals.

[6] It is not necessary to define $(\lambda_{i1}, \lambda_{i2}) \mapsto \lambda_i \in \mathbb{R}$ for other combinations of λ_{i1} and λ_{i2}.

The potential outcome of unit i in period t is $Y_{it}(\lambda)$, where $\lambda = (\lambda_1, ..., \lambda)$ is the treatment status of all units. Let $g = (g_1, ..., g_n)$ be the *group assignment vector*. The potential outcome is also an observed outcome when $\lambda = g$, that is,

$$Y_{it}^{\text{obs}} = Y_{it}(g).$$

The average treatment effect among the treated,

$$ATT = \mathbb{E}[Y_{i2}(g) - Y_{i2}(0)|g_i = 1], \tag{3.1}$$

is the period 2 difference between the average outcome among the treated under treatment and the average potential outcome under the counterfactual that no unit is treated. The main challenge in identifying ATT is that the potential outcome $Y_{i2}(0)$ is not observed by the empiricist.

Under some assumptions, it is well-known that ATT is identified as

$$DiD = \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 0], \tag{3.2}$$

since each term on the right hand side of (3.2) is observable. Note that (3.2) is the "difference-in-differences" of population means, or the *population DiD*. As exposited in Roth et al. (2023), the identification assumptions are parallel trends, no anticipatory effects and SUTVA.

The parallel trends assumption asserts that in the absence of treatment, both groups would have experienced the same outcome evolution, on average. The "no anticipatory effects" assumption says that period 1 potential outcomes do not depend on treatment status in period 2. The no interference component of SUTVA means that unit $i's$ potential outcomes do not depend on the treatment status of other units.[7] In the formal statement of the assumptions below, $\lambda_{-i} = (\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_n)$ is the treatment status of all units but i:

(AS1) *Parallel Trends.* $\mathbb{E}[Y_{i2}(0) - Y_{i1}(0)|g_i = 1] = \mathbb{E}[Y_{i2}(0) - Y_{i1}(0)|g_i = 0]$

(AS2) *No anticipatory effects.* $Y_{i1}(0, \lambda_{-i}) = Y_{i1}(1, \lambda_{-i})$ for all i and all λ_{-i}.

(AS3) *SUTVA (no interference).* For all i and $t = 1, 2$, $Y_{it}(\lambda_i, \lambda_{-i}) = Y_{it}(\lambda_i, \lambda'_{-i})$ for any $\lambda_{-i} \neq \lambda'_{-i}$.

If we drop SUTVA but maintain (AS1)-(AS2), then the causal effect identified by (3.2) is the marginal average treatment effect among the treated with spillovers, or MATTS:

[7] The other component of SUTVA, that there are no hidden variations of treatments, is maintained throughout. See Imbens and Rubin (2015) for more on SUTVA.

$$MATTS(g) = \underbrace{\mathbb{E}[Y_{i2}(g) - Y_{i2}(0)|g_i = 1]}_{ATTS} - \underbrace{\mathbb{E}[Y_{i2}(g) - Y_{i2}(0)|g_i = 0]}_{ASU} \quad (3.3)$$

MATTS is the difference between the average treatment effect among the treated with spillovers (ATTS) and the average spillover effect among the untreated (ASU). Note that if SUTVA is a maintained assumption, then $ASU = 0$ since $Y_{i2}(g) = Y_{i2}(0)$ for all i such that $g_i = 0$. In this case, MATTS and ATT are identical.

To see that (3.2) identifies MATTS when the no interference component of SUTVA is dropped, note that the no anticipatory effects assumption (AS2) allows us to write

$$\mathbb{E}[Y_{i1}(0)|g_i = 1] = \mathbb{E}[Y_{i1}(g)|g_i = 1] \text{ and } \mathbb{E}[Y_{i1}(0)|g_i = 0] = \mathbb{E}[Y_{i1}(g)|g_i = 0].$$

Make this substitution into the parallel trends assumption (AS1) and re-arrange to get

$$\mathbb{E}[Y_{i2}(0)|g_i = 0] - \mathbb{E}[Y_{i2}(0)|g_i = 1] = \mathbb{E}[Y_{i1}(g)|g_i = 0] - \mathbb{E}[Y_{i1}(g)|g_i = 1].$$

That is, the period 2 unobservable change in average outcomes under no treatment between those assigned to treatment and those not assigned to treatment equals the initial (i.e., period 1) observed difference in average outcomes.

Now re-arrange the right hand side of (3.3), substitute in the last result, and re-arrange a final time to get

$$\begin{aligned} MATTS(g) &= (\mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0]) + (\mathbb{E}[Y_{i2}(0)|g_i = 0] - \mathbb{E}[Y_{i2}(0)|g_i = 1]) \\ &= (\mathbb{E}[Y_{i2}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g)|g_i = 0]) + (\mathbb{E}[Y_{i1}(g)|g_i = 0] - \mathbb{E}[Y_{i1}(g)|g_i = 1]) \\ &= \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 1] - \mathbb{E}[Y_{i2}(g) - Y_{i1}(g)|g_i = 0]. \end{aligned}$$

In other words, we have shown that for a given group assignment g, MATTS is identified as the population DiD.

In summary, the population DiD identifies MATTS when the no interference component of SUTVA is dropped. When SUTVA is fully maintained, MATTS simplifies to the ATT. In either case, the population DiD is easily estimated using its sample analog.

4 The DiD Comparative Statics Problem

In this section we develop analytical restrictions on the sign of the causal estimands identified in the previous section. Our main objective is to restrict the sign of MATTS. But we are also intrinsically interested in the sign of ATTS and other comparative statics results. We provide these when possible.

With spillovers, (total) causal effects depend on the the size and composition of the treatment groups. Moreover, in practice the received treatment can differ from the intended treatment. Thus, we focus on sign predictions that are valid for every possible treatment group and for any treatment that has a positive direct impact on outcomes. In this way, the validity of any given statistical test of the sign restriction will not depend on the identity of the treatment group or whether the treated unit received the treatment in the prescribed amount.

The following framework is standard except for the way that treatments and treatment groups are handled.[8] There are n continuously differentiable functions $f^i(y;\lambda)$ for $i = 1, .., n$ where $y = (y_1, ..., y_n)$ is a vector of endogenous outcome variables and $\lambda = (\lambda_1, ..., \lambda_n)$ is a vector of exogenous parameters. We assume $y \in \mathcal{Y} \subset \mathbb{R}^n$ and $\lambda \in \mathit{\Lambda} \subset \mathbb{R}^n$, where Y and $\mathit{\Lambda}$ are open sets.

As an example, each function f^i may represent the marginal profit of firm i, y_i its output, and λ_i the unit tax applied to firm i. The equation $f^i(y;\lambda) = 0$ would be firm $i's$ first order condition for profit maximization.

Given $\lambda = \bar{\lambda}$, an *equilibrium* $\bar{y}$ is a solution to the system of equations:

$$\begin{aligned} f^1(y;\bar{\lambda}) &= 0 \\ &\vdots \\ f^n(y;\bar{\lambda}) &= 0. \end{aligned} \tag{4.1}$$

More compactly, let $f = (f^1, ..., f^n)$. Then, in equilibrium, $f(\bar{y};\bar{\lambda}) = 0$.

To describe the equilibrium effect of a change in the parameters λ among the treated, we need a way to select which parameters are changing to accommodate different treatment groups. To simplify, assume $\frac{\partial f^i}{\partial \lambda_j} = 0$ for all $j \neq i$ and $f^i_\lambda \equiv \frac{\partial f^i}{\partial \lambda_i} \neq 0$. This means, for example, that firm i's output is directly affected by a tax on firm i, but not by a tax on firm j.[9] Letting I denote the $n \times n$ identity matrix, put $G = Ig$ as the diagonal matrix whose main diagonal is the group assignment vector.[10] Let $D_\lambda f(\bar{y};\bar{\lambda})$ be the $n \times 1$ vector with typical element f^i_λ. Then $GD_\lambda f(\bar{y};\bar{\lambda})$ is the $n \times 1$ vector of *direct treatment effects*.

The Jacobian of f, $D_y f(y;\lambda)$, is the $n \times n$ matrix of partial derivatives, $f^i_j \equiv \frac{\partial f^i}{\partial y_j}$.

[8]In the comparative statics literature, "treatments" are often called "shocks."

[9]Firm i is indirectly affected by the tax on firm j if $\frac{\partial f^i}{\partial y_j} \neq 0$.

[10]Recall that $g = (g_1, ..., g_n)$, where g_i is an indicator equal to 1 if unit i is in the treated group and 0 otherwise.

Let $Dy(\bar{\lambda})$ denote the vector of *equilibrium treatment effects* whose i^{th} element is $\frac{d\bar{y}_i}{d\lambda} \equiv \sum_{j:g_j=1} \frac{d\bar{y}_i}{d\lambda_j}$. Then, by the IFT,[11]

$$Dy(\bar{\lambda}) = -[D_y f(\bar{y}; \bar{\lambda})]^{-1} G D_\lambda f(\bar{y}; \bar{\lambda}). \tag{4.2}$$

We are now able to define MATTS, ATTS, and ASU. Let n_t and n_u be the number of units in the treated and untreated groups, respectively, in the population. Note that $n_t + n_u = n$. Let δ_{ij} be the typical element of $-[D_y f(\bar{y}; \bar{\lambda})]^{-1}$. Then from (4.2) it follows that the average treatment effect on the treated with spillovers (ATTS) and the average spillover effect on the untreated (ASU) are, respectively,

$$ATTS = \frac{1}{n_t} \sum_{r:g_r=1} \sum_{s:g_s=0} \delta_{rs} f_\lambda^s \text{ and}$$

$$ASU = \begin{cases} \frac{1}{n_c} \sum_{r:g_r=0} \sum_{s:g_s=1} \delta_{rs} f_\lambda^s & \text{if } n_t < n \\ 0 & \text{if } n_t = n. \end{cases}$$

If $n_t = n$, then the whole population is treated so ASU is undefined, but for technical reasons we set it equal to zero. Then if $n_t < n$,

$$\begin{aligned} MATTS &= ATTS - ASU \\ &= \frac{1}{n_t} \sum_{r:g_r=1} \sum_{s:g_s=0} \delta_{rs} f_\lambda^s - \frac{1}{n_c} \sum_{r:g_r=0} \sum_{s:g_s=1} \delta_{rs} f_\lambda^s. \end{aligned} \tag{4.3}$$

And if $n_t = n$, $MATTS = ATTS$.

Example 1. We illustrate the model for $n = 3$. By (4.2), the total equilibrium effects of treatment are

$$\begin{bmatrix} \frac{d\bar{y}_1}{d\lambda} \\ \frac{d\bar{y}_2}{d\lambda} \\ \frac{d\bar{y}_3}{d\lambda} \end{bmatrix} = \begin{bmatrix} \delta_{11} f_\lambda^1 g_1 + \delta_{12} f_\lambda^2 g_2 + \delta_{13} f_\lambda^3 g_3 \\ \delta_{21} f_\lambda^1 g_1 + \delta_{22} f_\lambda^2 g_2 + \delta_{23} f_\lambda^3 g_3 \\ \delta_{31} f_\lambda^1 g_1 + \delta_{32} f_\lambda^2 g_2 + \delta_{33} f_\lambda^3 g_3 \end{bmatrix}.$$

Suppose only units 1 and 2 are treated, or $g = (1, 1, 0)$. Then

$$\begin{aligned} ATTS &= \frac{1}{2} \left\{ \left(\delta_{11} f_\lambda^1 + \delta_{12} f_\lambda^2 \right) + \left(\delta_{21} f_\lambda^1 + \delta_{22} f_\lambda^2 \right) \right\} \\ &= \frac{1}{2} \left\{ \left(\delta_{11} + \delta_{21} \right) f_\lambda^1 + \left(\delta_{22} + \delta_{12} \right) f_\lambda^2 \right\}, \\ ASU &= \delta_{31} f_\lambda^1 + \delta_{32} f_\lambda^2, \end{aligned}$$

[11] Assuming $\det -[D_y f(\bar{y}; \bar{\lambda})]^{-1} \neq 0$.

and

$$MATTS = \left[\frac{1}{2}(\delta_{11} + \delta_{21}) - \delta_{31}\right] f_\lambda^1 + \left[\frac{1}{2}(\delta_{22} + \delta_{12}) - \delta_{32}\right] f_\lambda^2.$$

Analogous formulas can be written for the other non-zero treatment vectors: $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$.

5 Mathematical Preliminaries

We say that the real variable x is *positive* if $x \geq 0$ and *strictly positive* if $x > 0$. Similarly, x is *negative* if $x \leq 0$ and *strictly negative* if $x < 0$.

Consider the $n \times n$ real matrix $A = (a_{ij})$. A is a *P-matrix* (P_0-matrix) if all of its principal minors are strictly positive (positive). A is a *Z-matrix* if all of its off-diagonal terms are negative. A is a *M-matrix* if it is a nonsingular Z-matrix and it has a positive inverse, $A^{-1} \geq 0$. The terms *B-matrix, B_0-matrix, B-matrix by columns, and B_0-matrix by columns* were defined in Section 2. A is *strictly diagonally dominant (SDD)* if, for $i = 1, ..., n$,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

The diagonal entry of an SDD matrix is larger in magnitude than the sum of the absolute values of the elements in the same row.

It is well-known that an *SDD* matrix with a strictly positive diagonal is a P-matrix with a strictly positive determinant. A B-matrix (B_0-matrix) also has a strictly positive (positive) determinant, and no weaker linear condition exists under which A has this property (Carnicer, Goodman, and Peña, 1999). A B-matrix (B_0-matrix) is also a P-matrix (P_0-matrix) (Peña, 2001). A B-matrix (B_0-matrix) has a strictly positive (positive) diagonal, and, in fact, $a_{ii} > r_i^+$ ($a_{ii} \geq r_i^+$) for all i (Peña, 2001).[12]

6 Conditions on Equilibrium Spillover Effects

We begin by seeking conditions on the inverse of the negated Jacobian, $[-D_y f(\bar{y}; \bar{\lambda})]^{-1} = (\delta_{ij})$, under which we can sign MATTS and other quantities of interest. The elements of the inverse (δ_{ij}) capture equilibrium spillover effects between units i and j.

Our first result can be illustrated in the $n = 3$ case from Example 1. When $g = (1, 1, 0)$, $ATTS \geq 0$ whenever $f_\lambda^1, f_\lambda^2 > 0$ if and only if $\delta_{11} + \delta_{21} \geq 0$ and $\delta_{21} + \delta_{22} \geq 0$. Note that the

[12] For a B-matrix, $a_{ii} > nr_i^+ - \sum_{j \neq i} a_{ij} \geq nr_i^+ - (n-1)r_i^+ = r_i^+ \geq 0$. For a B_0-matrix the first inequality is weak, but otherwise the argument is the same.

latter two inequalities involve the partial sums of columns 1 and 2 of the negated Jacobian inverse. Each partial column sum includes its diagonal term, δ_{11} and δ_{22}. These are examples of *diagonally-centered partial column sums.*

Similarly, $ASU \geq 0$ whenever $f_\lambda^1, f_\lambda^2 > 0$ if and only if $\delta_{31} \geq 0$ and $\delta_{32} \geq 0$. The latter terms are the elements that were excluded from the diagonally-centered partial column sums of columns 1 and 2. These are examples of *non-diagonally-centered partial column sums.*

Finally, $MATTS \geq 0$ whenever $f_\lambda^1, f_\lambda^2 > 0$ if and only if the average of the terms included in the diagonally-centered partial column sum exceeds the average of the terms excluded from that sum, by column. That is, $\frac{1}{2}(\delta_{11} + \delta_{21}) \geq \delta_{31}$ and $\frac{1}{2}(\delta_{12} + \delta_{22}) \geq \delta_{32}$.

This illustrates the vital role played by the partial column sums of the inverse Jacobian in signing MATTS, ATTS, and ASU. Theorem 1 generalizes this argument to allow for any group assignment. Define $\Gamma = \{g = (g_1, ..., g_n) \in \mathbb{R}^n | g_i \in \{0, 1\} \ \forall i \text{ and } g_i = 1 \text{ for some } i\}$ as the set of all possible non-zero group assignment vectors. We suppress arguments in the following results, but they are all equilibrium results.

Theorem 1. *Suppose $D_y f(\bar{y}; \bar{\lambda})$ is nonsingular.*

1. $ATTS \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all $s : g_s = 1$ and all $g \in \Gamma$. Moreover $ATTS > 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff, for any $g \in \Gamma$, $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all $s : g_s = 1$ and $\sum_{r:g_r=1} \delta_{rs} > 0$ for some $s : g_s = 1$.

2. $ASU \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff $\sum_{r:g_r=0} \delta_{rs} \geq 0$ for all $s : g_s = 1$ and all $g \in \Gamma$. Moreover, if $n_t < n$, then $ASU > 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff, for any $g \in \Gamma$, $\sum_{r:g_r=0} \delta_{rs} \geq 0$ for all $s : g_s = 1$ and $\sum_{r:g_r=0} \delta_{rs} > 0$ for some $s : g_s = 1$.

3. $MATTS \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff $\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \geq \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}$ for all $s : g_s = 1$ and all $g \in \Gamma$ such that $n_t < n$, and $\frac{1}{n} \sum_{r=1}^n \delta_{rs} \geq 0$ for $s = 1, ..., n$. Moreover, $MATTS > 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff, for any $g \in \Gamma$, $\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \geq \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}$ for all $s : g_s = 1$ (with strict inequality for some $s : g_s = 1$) and $\frac{1}{n} \sum_{r=1}^n \delta_{rs} \geq 0$ for $s = 1, ..., n$ (with strict inequality for some $s = 1, ..., n$).

Remark 1. A more general result holds. In addition to the necessary and sufficient result in part 1, we can also write

- $ATTS \leq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff $\sum_{r:g_r=1} \delta_{rs} \leq 0$ for all $s : g_s = 1$ and all $g \in \Gamma$.

- $ATTS \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f < 0$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all s :$g_s = 1$ and all $g \in \Gamma$.

- $ATTS \leq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f < 0$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all s :$g_s = 1$ and all $g \in \Gamma$.

The statement concerning $ATTS > 0$ whenever $D_\lambda f > 0$ can be similarly generalized. The same holds for parts 2 and 3 of the theorem.

In words, Theorem 1 says that ATTS (ASU) is positive for any non-trivial group assignment vector whenever direct treatment effects are strictly positive among the treated if, and only if, every (non-)diagonally-centered partial column sum of the Jacobian inverse is positive. Moreover, ATTS (ASU) is strictly positive if and only if every (non-)diagonally-centered partial column sum of the Jacobian inverse is positive, with at least one sum being strictly positive for every nontrivial treatment group.

While Theorem 1 provides nice necessary and sufficient conditions such that ATTS and ASU are positive, it says that MATTS is positive iff the average of any set of column entries which includes the diagonal term is larger than the average of the remaining column entries. This is harder to conceptualize, but there is a profitable simplification.

We illustrate the idea in the $n = 3$ case. Theorem 1 says that MATTS is positive for any group assignment $g \in \Gamma$ whenever $D_\lambda f > 0$ if and only if the following 4 inequalities hold for each $i = 1, 2, 3$ and $j \neq k \neq i$:

$$\delta_{ii} \geq \frac{1}{2}\left(\delta_{ji} + \delta_{ki}\right) \tag{6.1}$$

$$\frac{1}{2}\left(\delta_{ii} + \delta_{ji}\right) \geq \delta_{ki} \tag{6.2}$$

$$\frac{1}{2}\left(\delta_{ii} + \delta_{ki}\right) \geq \delta_{ji} \tag{6.3}$$

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \geq 0. \tag{6.4}$$

Those four inequalities are equivalent to these three:

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \geq 3\delta_{ji} \tag{6.5}$$

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \geq 3\delta_{ki} \tag{6.6}$$

$$\delta_{ii} + \delta_{ji} + \delta_{ki} \geq 0. \tag{6.7}$$

Inequalities (6.5)-(6.7) are the same as (6.2)-(6.4). To get (6.1), add (6.5) and (6.6), and then rearrange:

$$2(\delta_{ii} + \delta_{ji} + \delta_{ki}) \geq 3(\delta_{ji} + \delta_{ki})$$
$$\delta_{ii} \geq \frac{1}{2}(\delta_{ji} + \delta_{ki}).$$

Notice that inequalities (6.5)-(6.7) are equivalent to saying that the inverse of the negated Jacobian, $-[D_\lambda f]^{-1}$, is a B_0-matrix by columns! Our next result says that this result holds for arbitrary $n < \infty$.

Theorem 2. *Suppose $D_y f(\bar{y}; \bar{\lambda})$ is nonsingular. $MATTS \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ iff $-[D_y f]^{-1}$ is a B_0-matrix by columns. Moreover, $MATTS > 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ if $-[D_y f]^{-1}$ is a B-matrix by columns.*

Remark 2. For both results, the sign of $MATTS$ is reversed if the sign of $D_\lambda f$ is reversed.

To interpret this result, we can think of the elements of the inverse of the negated Jacobian, δ_{ij}, as the equilibrium effect of a one unit increase in unit j's outcome on unit $i's$ outcome. If the negated Jacobian is a B_0-matrix by columns, this means that the equilibrium effect of any unit j on unit i's outcome cannot be larger than the average equilibrium effect on unit i, where the average equilibrium effect includes the own equilibrium effect δ_{ii}. This condition rules out direct spillover effects which accumulate into outlier equilibrium effects.

Theorem 2 is a fascinating result. First it identifies a well-known class of matrices, B-matrices, which characterize the Jacobian inverse such that MATTS is positive. This means that to determine the type of direct spillover effects under which MATTS is (strictly) positive, we can focus attention on the class of matrices whose transposed inverse is a B_0-matrix (B-matrix). This task is taken up in the next section.

In addition, the characterization of MATTS provided in part 3 of Theorem 1 relies on significantly more inequalities than the characterization in Theorem 2. For a given n, part 3 of Theorem 1 requires us to check

$$\sum_{j=0}^{n-1} \binom{n-1}{j}$$

inequalities per column while the B_0-matrix property requires only n.[13] The remaining in-

[13]For each column, the inequalities in part (3) of Theorem 1 involve every difference between the diagonally-centered partial column sum and the sum of the remaining column entries. Thus, the number of inequalities to check is the same as the number diagonally-centered partial column sums. Each of these sums includes the diagonal element, to which we add between 0 and $n-1$ off-diagonal elements. If j off-diagonal elements are included, there are $n-1$ choose j inequalities.

equalities in part 3 of Theorem 1 are redundant. To give a sense of the scale of simplification, note that if $n = 15$, then part 3 of Theorem 1 involves checking 16,384 inequalities per column for a total of 245,760 inequalities; Theorem 2 reduces this to 15 per column for a total of 225. If $n = 25$, the totals are over 419 million compared to just 625.

Finally, three useful corollaries easily obtain as a consequence of the B_0-matrix characterization. First, if MATTS is always positive, then ATTS is also always positive. On first impression, this is surprising since MATTS is the difference between ATTS and ASU. But any result that declares MATTS to be positive for any group assignment whenever $D_\lambda f > 0$ must also be true when ASU is zero. Second, then if there is a single treated unit, then the impact of treatment on this unit is positive and larger in magnitude than the impact on any other unit. Finally, treatment increases the total outcome. That is, if $Y \equiv \sum_{i=1}^{n} y_i$, then $\frac{d\bar{Y}}{d\lambda} \equiv \sum_{i=1}^{n} \frac{d\bar{y}_i}{d\lambda} \geq 0$. To avoid excessive repetition, subsequent theorems refer to these three results as *"the results of Corollary 1."*

Corollary 1. *Suppose $-D_y f$ is nonsingular. If $-[D_y f]^{-1}$ is a B-matrix (B_0-matrix) by columns, then*

1. *$ATTS > (\geq)\ 0$ for every group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$,*
2. *$\frac{d\bar{Y}}{d\lambda} > (\geq)\ 0$ for every group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$ and*
3. *if, in addition, unit i is the only treated unit ($n_t = 1$), $\frac{d\bar{y}_i}{d\lambda} > (\geq) \left|\frac{d\bar{y}_j}{d\lambda}\right|$ for all $j \neq i$ whenever $D_\lambda f > 0$.*

Proof. We prove the result for B-matrices. The result for B_0-matrices is analogous.

(1) By the definition of a B-matrix by columns, $\sum_i \delta_{ij} > nc_j^+$ for $j = 1, ..., n$. Let $H = \{h \in \mathbb{N} | 1 \leq h \leq n \text{ and } \delta_{hj} < 0\}$. If $g_s = 1$, then $\sum_{r:g_r=1} \delta_{rs} \leq 0$ only if there are some terms δ_{rs} in the sum such that $r \in H$. But by Proposition 2.4 in Peña (2001), $\delta_{jj} > \sum_{h \in H} |\delta_{hj}|$.[14] It follows that for all $g \in \Gamma$, $\sum_{r:g_r=1} \delta_{rs} > 0$ for all $s : g_s = 1$. Thus, $ATTS > 0$ by Theorem 1.

(2) By equation (4.2), $\frac{d\bar{Y}}{d\lambda} = \sum_i \sum_j \delta_{ij} f_\lambda^j g_j = \sum_j \left(\sum_i \delta_{ij}\right) f_\lambda^j g_j$. Note that $\sum_i \delta_{ij} > 0$ since $-[D_y f]^{-1}$ is a B-matrix by columns. This proves the result.

(3) If $g_i = 1$ and $g_j = 0$ for all $j \neq i$, then by equation (4.2) we have $\frac{d\bar{y}_i}{d\lambda} = \delta_{ii} f_\lambda^i$ and $\frac{d\bar{y}_j}{d\lambda} = \delta_{ji} f_\lambda^j$ for all $j \neq i$, so it suffices to prove $\delta_{ii} > |\delta_{ji}|$ for all $j \neq i$. But this follows from Proposition 2.4 in Peña (2001). ☐

[14] $\delta_{jj} > nc_j^+ - \sum_{i \neq j} \delta_{ij} = nc_j^+ - \sum_{h \neq j, h \neq H} \delta_{hj} + \sum_{h \in H} |\delta_{hj}|$.

7 Conditions on Direct Spillover Effects

Theorems 1-2 are especially useful if the Jacobian can be inverted in a nice closed form. However, there are many economic problems where this is not feasible. In this section we concentrate on conditions on the (noninverted) Jacobian under which MATTS and ATTS are positive. Put another way, rather than finding conditions on the equilibrium spillover effects, δ_{ij}, we now focus on finding conditions on the direct spillover effects, f^i_j.

By Theorem 2 and Corollary 1 it is sufficient to find conditions on the negated Jacobian such that its inverse is a B-matrix by columns. This is a challenging problem in general, but we are able to make headway in some economically relevant special cases. Similar to Christensen (2019), the overall theme of the findings is that a trade-off exists between the heterogeneity and magnitude of spillovers to guarantee intuitive comparative statics results.

7.1 Anonymous-By-Unit Spillovers

A great deal of structure emerges if spillover effects are *anonymous-by-unit*, meaning that $f^i_i = \alpha_i$ and $f^i_j = \beta_i$ for all $j \neq i$ and all $i = 1, ..., n$. In this case, a unit increase in y_j has the same effect on y_i as a unit increase in y_k, for any $j \neq k \neq i$. In matrix form,

$$D_y f(\bar{y}, \bar{\lambda}) = \begin{bmatrix} \alpha_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & \alpha_2 & \cdots & \beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_n & \cdots & \alpha_n \end{bmatrix}.$$

This case arises when the component functions take the form $f^i(y; \lambda) = f^i(y_i, \sum_{j \neq i} y_j; \lambda)$ such as in Cournot competition where a firm's demand, which can be different for each firm, depends on rivals' outputs only through their sum.

Dixit (1986) derives closed form comparative statics formulae for this case. To sign the comparative statics, he assumes that $\alpha_i < 0$ and the matrix $-D_y f(\bar{y}, \bar{\lambda})$ is SDD: for $i = 1, ..., n$, $|-\alpha_i| > (n-1)|-\beta_i|$, or,

$$\begin{cases} \alpha_i < (n-1)\beta_i & \text{if } \beta_i \leq 0 \\ \alpha_i < -(n-1)\beta_i & \text{if } \beta_i > 0. \end{cases}$$

Under these assumptions he derives the following results on equilibrium variables for the case when only one unit is treated.

Proposition 1 (Dixit, 1986). *Suppose spillover effects are anonymous-by-unit, that* $-D_y f(\bar{y}, \bar{\lambda})$

is SDD, and that only one unit is treated ($n_t = 1$). Then for any vector of direct treatment effects $D_\lambda f > 0$,

1. $\frac{d\bar{y}_i}{d\lambda} > 0$.

2. $\frac{d\bar{y}_j}{d\lambda}$ is negative or positive for $j \neq i$ as β_j is negative or positive.

3. $\frac{d\bar{Y}}{d\lambda} > 0$.

Christensen (2019) showed that parts (1) and (3) of Proposition 1 hold under the weaker assumption that $-D_y f(\bar{y}, \bar{\lambda})$ is a B-matrix, that is, when $-\alpha_i - (n-1)\beta_i > n \max\{0, -\beta_i\}$ for $i = 1, ..., n$, or

$$\begin{cases} \alpha_i < \beta_i & \text{if } \beta_i \leq 0 \\ \alpha_i < -(n-1)\beta_i & \text{if } \beta_i > 0. \end{cases}$$

To see that this is a weaker restriction than SDD when spillovers are anonymous-by-unit, note that $-D_y f$ is a B-matrix if it is SDD. On the other hand, the following is an example of a B-matrix with a positive diagonal that is not SDD:

$$\begin{bmatrix} 3 & 2.5 & 2.5 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Remarkably, by direct computation we can also sign MATTS, ATTS, and ASU for any treatment group, not just those with a single treated unit. To this end, let

$$\Gamma = 1 + \sum_{i=1}^{n} \beta_i / (\alpha_i - \beta_i).$$

Per Dixit (1986) we have

$$\frac{d\bar{y}_i}{d\lambda} = -\frac{g_i f_\lambda^i}{\alpha_i - \beta_i} + \frac{\beta_i}{\Gamma(\alpha_i - \beta_i)} \sum_{j=1}^{n} \frac{g_j f_\lambda^j}{\alpha_j - \beta_j} \text{ and} \tag{7.1}$$

$$\frac{dY}{d\lambda} = -\frac{1}{\Gamma} \sum_{i=1}^{n} \frac{g_i f_\lambda^i}{\alpha_i - \beta_i}. \tag{7.2}$$

If $-D_y f$ is a B-matrix and $\beta_i > 0$ then

$$-\frac{1}{n} = -\frac{\beta_i}{n\beta_i} < \frac{\beta_i}{\alpha_i - \beta_i} < 0. \tag{7.3}$$

If $\beta_i \leq 0$ then $\beta_i/(\alpha_i - \beta_i) \geq 0$. From these two facts we have $\Gamma = 1 + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i - \beta_i} > 0$. It follows that $\frac{d\bar{Y}}{d\lambda} > 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$.

Turning to average impacts, note that

$$\begin{aligned}
ATTS &= \frac{1}{n_t}\left[-\sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j} + \sum_{k:g_k=1} \frac{\beta_k/(\alpha_k - \beta_k)}{\Gamma} \sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j}\right] \\
&= -\frac{1}{n_t}\left[\sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j}\left(1 - \sum_{k:g_k=1} \frac{\beta_k/(\alpha_k - \beta_k)}{\Gamma}\right)\right], \\
ASU &= \frac{1}{n_u} \sum_{k:g_k=0} \frac{\beta_k/(\alpha_k - \beta_k)}{\Gamma} \sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j}, \text{ and} \\
MATTS &= -\sum_{j:g_j=1} \frac{f_\lambda^j}{\alpha_j - \beta_j} \left\{ \begin{array}{c} \frac{1}{n_t}\left(1 - \sum_{k:g_k=1} \frac{\beta_k/(\alpha_k - \beta_k)}{\Gamma}\right) \\ + \frac{1}{n_u} \sum_{k:g_k=0} \frac{\beta_k/(\alpha_k - \beta_k)}{\Gamma} \end{array} \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\Gamma - \sum_{k:g_k=1} \frac{\beta_k}{\alpha_k - \beta_k} &= 1 + \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} \\
&> 1 - \frac{n_u}{n} \\
&\geq 0,
\end{aligned}$$

it follows that ATTS> 0. The sign of ASU is ambiguous in general, but it is strictly positive (strictly negative) if $\beta_k \geq (\leq) 0$ for all k such that $g_k = 0$ with strict inequality for some k such that $g_k = 0$. To sign MATTS, observe that

$$\begin{aligned}
\Gamma - \sum_{j:g_j=1} \frac{\beta_j}{\alpha_j - \beta_j} + \frac{n_t}{n_u} \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} &= 1 + \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} + \frac{n_t}{n_u} \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} \\
&= 1 + \frac{n}{n_u} \sum_{k:g_k=0} \frac{\beta_k}{\alpha_k - \beta_k} \\
&> 1 - \frac{n}{n_u} \frac{n_u}{n} \\
&= 0.
\end{aligned}$$

It follows that MATTS> 0. These findings are summarized in the theorem below.

Theorem 3. *Suppose spillovers are anonymous-by-unit and that $-D_y f(\bar{y}, \bar{\lambda})$ is a B-matrix. Then for any group assignment g and any vector of treatment effects $D_\lambda f > 0$,*

1. *$MATTS > 0$,*

2. *the results of Corollary 1 apply, and*

3. $$\begin{cases} ASU > 0 & \text{if } \beta_k \geq 0 \text{ for all } k : g_k = 0 \text{ and } \beta_k > 0 \text{ for some } k : g_k = 0 \\ ASU < 0 & \text{if } \beta_k \leq 0 \text{ for all } k : g_k = 0 \text{ and } \beta_k > 0 \text{ for some } k : g_k = 0. \end{cases}$$

Is there any weaker condition which guarantees that MATTS is strictly positive when spillovers are anonymous-by-unit? No, not if $\alpha_i \neq \beta_i$. In this case $-D_y f$ is a B-matrix if it is a B_0-matrix, and $-D_y f$ is a B-matrix if and only if $[-D_y f]^{-1}$ is a B-matrix by columns. In view of Theorem 2, this means that the B-matrix condition is necessary and sufficient for MATTS to be positive.

Lemma 1. *Suppose for all $i = 1, ..., n$, $f^i_i = \alpha_i \neq 0$ and $f^i_j = \beta_i \neq \alpha_i$ for all $j \neq i$. Then $-D_y f(\bar{y}; \bar{\lambda})$ is a B-matrix (by rows) if and only if $-[D_y f(\bar{y}; \bar{\lambda})]^{-1}$ a B-matrix by columns.*

Sufficiency follows from Theorem 2 since we have shown that MATTS>0 when $-D_y f$ is a B-matrix. We prove necessity and provide an alternate sufficiency proof in the Appendix. We apply Lemma 1 as follows:

Corollary 2. *If spillover effects are anonymous-by-unit and $\alpha_i \neq \beta_i$ $\forall i$, then MATTS> 0 for any group assignment g whenever $D_\lambda f > 0$ if and only if $-D_\lambda f(\bar{y}, \bar{\lambda})$ is a B-matrix.*

Proof. This follows from Lemma 1 and Theorem 2. □

7.2 Positive Spillovers

Assume spillovers are positive, or $f^i_j \geq 0$ for all $i \neq j$. As illustrated in the next section, this case arises, for example, in games with strategic complements. If $-D_y f$ has a strictly positive diagonal then it is a Z-matrix by definition. If it is also an M-matrix it has a positive inverse, and thus the outcome of every unit weakly increases; hence, ATTS and ASU are positive. In fact, we can also show that this is a necessary condition. The sign of MATTS is ambiguous in general, but the following result gives conditions under which MATTS is strictly positive.

Theorem 4. *Suppose for $i = 1, ..., n$, $f^i_j \geq 0$ for all $i \neq j$ and $f^i_i < 0$.*

1. For all $i = 1, ..., n$, $\frac{d\bar{y}_i}{d\lambda} \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f \geq 0$ if and only if $-D_y f(\bar{y}; \bar{\lambda})$ is an M-matrix. It follows that $ATTS \geq 0$ and $ASU \geq 0$.

2. Suppose $-f_i^i > (n-1)\max_{j\neq i}\{f_j^i\}$ for all $i = 1, ..., n$. Then $MATTS > 0$ and $ASU \geq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$. Moreover, the results of Corollary 1 apply.

The condition in part 2 of Theorem 4 implies that $-D_y f(\bar{y}; \bar{\lambda})$ is an M-matrix. To see this, note that

$$|f_i^i| = -f_i^i > (n-1)\max_{j\neq i}\{f_j^i\} \geq \sum_{j\neq i} |f_j^i|$$

implies $-D_y f(\bar{y}; \bar{\lambda})$ is SDD with a positive diagonal, and is thus a P-matrix. It follows that $-D_y f(\bar{y}; \bar{\lambda})$ is also an M-matrix since a nonsingular Z-matrix that is also a P-matrix is an M-matrix (Plemmons, 1977).

Intuitively, when spillovers are positive, an increase in any unit's outcome (weakly) increases the outcomes of all other units. Thus, a treatment that increases the outcome of any unit(s) should increase the outcome of all units, provided that the equilibrium system is well-behaved. Consequently, we expect ATTS and ASU to be positive. The system is well-behaved if its Jacobian is an M-matrix, which can be thought of as a stability requirement (Plemmons, 1977; Christensen and Cornwell, 2018).

7.3 Strictly Negative Spillovers

We can also obtain strong results if spillovers are strictly negative, meaning $f_j^i < 0$ for all i, j. For then the negated Jacobian $-D_y f$ is a strictly positive matrix. If its inverse is an M-matrix, the terms on the main diagonal are strictly positive and the off-diagonal terms are positive. From the latter it follows that ASU is negative. In fact, $\frac{d\bar{y}_j}{d\lambda} \leq 0$ for any untreated unit. If, in addition, the inverse is SDD by columns, it follows that ATTS and MATTS are strictly positive, as claimed in Theorem 5 below. In fact, $\frac{d\bar{y}_j}{d\lambda} \geq 0$ for any treated unit. The result relies largely on Willoughby (1977) which provides tight sufficient conditions under which the inverse of a positive matrix is an M-matrix.

Lemma 2 (Willoughby, 1977)**.** *Suppose $-f_j^i > 0$ for all $i, j = 1, ..., n$. Assume $0 < y \leq x < 1$ and for $i \neq j$, $0 < y \leq f_j^i / f_i^i \leq x < 1$. Let the interpolation parameter, s, be defined by*

$$x^2 = sy + (1-s)y^2.$$

Further suppose that any of the following conditions is satisfied:

1. $n = 2$,

2. $x = y$, or

3. $n \geq 3$ *and* $s \leq \frac{1}{n-2}$.

Then $[-D_y f]^{-1}$ *exists and is a SDD (by rows and columns) M-matrix.*

Theorem 5. *Suppose the conditions of Lemma 2 are satisfied. Then for any group assignment* $g \in \Gamma$ *and* $D_\lambda f > 0$, $MATTS > 0$, $ASU \leq 0$, $\frac{d\bar{y}_j}{d\lambda} \leq 0$ *for any* j *such that* $g_j = 0$, *and* $\frac{d\bar{y}_j}{d\lambda} > 0$ *for any* j *such that* $g_j = 1$. *Moreover, the results of Corollary 1 apply.*

Intuitively, when spillovers are strictly negative, the direct spillover effect of an increase in a single unit's outcome decreases the outcomes of all other units. Thus, if a single unit receives a treatment with a strictly positive direct treatment effect, downward pressure is exerted on the outcomes of all other units. Consequently, we expect the treated unit's outcome to increase while untreated units' outcomes decrease. This in turn implies that MATTS is strictly positive. For a well-behaved system—as defined by the conditions in Lemma 2—this intuition extends to any treatment group. Note that these conditions are global in the sense that they constrain the heterogeneity of direct spillover effects of all units jointly. In contrast, Theorems 3 and 4 constrain the heterogeneity of direct spillover effects acting on a single unit. Put differently, Lemma 2 is a joint condition on all the off-diagonal terms of the negated Jacobian, whereas the other results are conditions which apply independently to each row of the matrix.

We finish this subsection with the technical observation that a Z-matrix with a strictly positive diagonal is a B-matrix by columns if, and only if, it is SDD by columns. This fact further clarifies the relationship between B-matrices and SDD matrices, as well as the the relationship between Theorem 5, Lemma 2, and Theorem 2.

Fact 1. *Let* $A = (a_{ij})$ *be an* $n \times n$ *real matrix with a strictly positive diagonal and negative off-diagonal terms: for* $i = 1, ..., n$, $a_{ii} > 0$ *and* $a_{ij} \leq 0$ *for all* $j \neq i$. *Then* A *is SDD by columns if and only if it is a* B*-matrix by columns.*

Proof. In this case $c_j^+ = 0$ for all j, so A is a B-matrix by columns if $\sum_{i=1}^n a_{ij} > 0$, or $a_{jj} > -\sum_{i \neq j} a_{ij}$. This is equivalent to $|a_{jj}| > \sum_{i \neq j} |a_{ij}|$. □

7.4 Small Spillover Effects

In this section we formalize the intuition that small spillovers should not be able to overcome the direct effect of treatment. To this end, Theorem 6 below provides a sufficient condition under which MATTS is strictly positive, which in turn implies that the results of Corollary 1 apply. Viewed through a linear algebra lens, Theorem 6 identifies a matrix whose inverse is a B-matrix by columns. The proof relies on the following Lemma.

Lemma 3. *Suppose A is an $n \times n$ real matrix and, for $i = 1, ..., n$,*

$$a_{ii} > (n-1) \sum_{j \neq i} |a_{ij}| .$$

Then A is a SDD (by rows) B-matrix.

Theorem 6. *Suppose for $i = 1, ..., n$,*

$$-f_i^i > (n-1)^2 \sum_{j \neq i} \left| -f_j^i \right| . \tag{7.4}$$

Then $MATTS > 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$. Moreover, the results of Corollary 1 apply.

8 Application: Taxes and Output

In this section we return to question of whether the profit maximization hypothesis can be tested via the comparative statics of taxation. We provide a simple but generalizable model supporting the summary discussion in the introduction which argued that this is not possible in general because of the sign reversal property of the population DiD. Then we apply Theorems 3-6 to identify assumptions on the competitive environment under which MATTS is strictly negative when any subset of firms is taxed, a testable prediction within the DiD framework. Environments in which MATTS is strictly negative after a tax include homogeneous Cournot competition and perfect competition, among others.

8.1 Imperfect Competition

Consider an imperfectly competitive population of n firms where each firm i selects output $y_i \geq 0$ to maximize profit π^i. Firm i's cost $c_i(y_i) - \gamma_i \lambda_i y_i$ depends on its output and a unit tax λ_i. Assume that production cost $c_i(y_i)$ is convex, $c_i''(y_i) \geq 0$ for all $y_i \geq 0$ and all i. The subscript on λ_i allows the tax to be firm-specific. The parameter $\gamma_i > 0$ allows for the actual tax treatment to differ in magnitude, but not the sign, from the intended treatment. Inverse demand for each firm is linear, $p_i(y) = a_i - \sum_{j \neq i j=1}^{n} b_{ij} y_j - \frac{1}{2} b_{ii} y_i$ with $b_{ii} > 0$ for all i.

We assume a unique interior equilibrium exists. The first order condition for each firm $i = 1, ..., n$ is

$$\frac{\partial \pi_i}{\partial y_i} = a_i - \sum_{j=1}^{n} b_{ij} y_j - c_i'(y_i) - \gamma_i \lambda_i = 0.$$

Equilibrium $\bar{y} = (\bar{y}_1, ..., \bar{y}_n)$ is a solution to this system of n equations. Define $f^i \equiv \frac{\partial \pi_i}{\partial y_i}$ to match the notation from Section 4. The negated Jacobian of the system is

$$-D_y f(\bar{y}; \bar{\lambda}) = \begin{bmatrix} b_{11} + c_1''(\bar{y}_1) & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & +c_2''(\bar{y}_2) \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} + c_n''(\bar{y}_n) \end{bmatrix}.$$

Notice that the terms of the negated Jacobian can be interpreted as the slope coefficients of firm demand, or as the change in marginal profit since $\frac{\partial^2 \pi_i}{\partial y_i \partial y_j} = -b_{ij}$. The latter interpretation generalizes so we use it. The vector of direct treatment effects is

$$D_\lambda f(\bar{y}; \bar{\lambda}) = \begin{bmatrix} -\gamma_1 \\ \vdots \\ -\gamma_n \end{bmatrix} < 0.$$

8.1.1 Illustration of the Sign Reversal Property

We first present an example to illustrate the sign reversal property of the population DiD when interpreted as the ATT. Let $n = 3$. Set $c_i(y_i) \equiv 0$ and $\gamma_i = 1$ for $i = 1, 2, 3$. Suppose

$$\begin{aligned} p_1(y) &= a - \frac{1}{6} y_1, \\ p_2(y) &= a - \frac{1}{2} y_2, \text{ and} \\ p_3(y) &= a + \frac{5}{4} y_1 + \frac{5}{4} y_2 - y_3. \end{aligned}$$

Firm 3's demand is complementary with firm 1 and 2's output, but firm 1 and firm 2's demand is independent of the others. These stark assumptions are designed to illustrate the mechanics of the sign reversal property.

Equilibrium is the solution to the system of first order conditions:

$$\begin{aligned} \pi_1^1(y; \gamma_1, \lambda_1) &= a - \frac{1}{3} y_1 - \lambda_1 = 0, \\ \pi_1^2(y; \gamma_2, \lambda_2) &= a - y_2 - \lambda_2 = 0, \text{ and} \\ \pi_1^3(y; \gamma_3, \lambda_3) &= a + \frac{5}{4} y_1 + \frac{5}{4} y_2 - 2y_3 - \lambda_3 = 0. \end{aligned} \tag{8.1}$$

The equilibrium quantities are

$$\bar{y}_1 = 3a - 3\lambda_1,\ \bar{y}_2 = a - \lambda_2,\ \text{and} \bar{y}_3 = 3a - \frac{15}{8}\lambda_1 - \frac{5}{8}\lambda_2 - \frac{1}{2}\lambda_3.$$

If firms 1 and 2 are treated while firm 3 is untreated, then

$$\frac{d\bar{y}_1}{d\lambda} = -3,\ \frac{d\bar{y}_2}{d\lambda} = -1,\ \text{and}\ \frac{d\bar{y}_3}{d\lambda} = -\frac{5}{2}.$$

Each firm reduces output in equilibrium, yet

$$MATTS = \frac{-3-1}{2} - \left(-\frac{5}{2}\right) = \frac{1}{2} > 0.$$

In practice, a researcher estimates the population DiD which in turn identifies MATTS. But if one were to interpret the population DiD as the ATT, then one would erroneously reject the hypothesis that the tax decreases output among the taxed, on average.

One may take issue in this example with the fact that in firm 3's demand function, the slope coefficients on firm 1 and 2's output ($-5/4$) is larger in magnitude than the slope coefficient on firm 3's output (1). There are two responses to this concern. First, this type of situation is ruled out in order to guarantee that MATTS is strictly negative. Second, we can extend this example to n firms where firms 1 to $n-1$ experience no spillover effects ($b_{ij} = 0$ for $i = 1, ..., n-1$ and $j \neq i$) and firm n experiences anonymous spillover effects ($b_{nj} = \beta_n < 0$ for all $j \neq n$). Then, by Corollary 2, MATTS is strictly negative if and only if $b_{nn} < -(n-1)\beta = (n-1)|\beta|$. Thus, whenever $n > 3$ we can have $|\beta| < b_{nn}$ and MATTS strictly positive.

8.1.2 Anonymous-by-unit Spillovers

Suppose spillovers are anonymous-by-unit so $b_{ij} = \beta_i$ for all $j \neq i$ and all $i = 1, ..., n$. It follows from Corollary 2 that MATTS is strictly negative when the unit tax increases for any nontrivial subset of firms if, and only if,

$$\begin{aligned} b_{ii} + c_i''(\bar{y}_i) &> \beta_i && \text{when } \beta_i > 0, \text{ and} \\ b_{ii} + c_i''(\bar{y}_i) &> -(n-1)\beta_i && \text{when } \beta_i \leq 0. \end{aligned}$$

By Theorem 3 and Proposition 1, these conditions also imply that ATTS is strictly negative, total output decreases, and if only one firm is taxed, then the taxed firm decreases its output by more than the output of any other firms changes while the the output of the untaxed firms decreases.

To understand the intuition behind the conditions, first suppose that $\beta_i > 0$ for all i. Then firm $i's$ output is a strategic substitute with rivals' output since $\frac{\partial \pi_i}{\partial y_i \partial y_j} = -\beta_i < 0$. Since the direct treatment effect of a tax on any firm is strictly negative, the direct spillover effect on a firm of any taxed rival is an increase in own output. But as long as marginal profit is affected most by changes in own output, equilibrium spillover effects will not overpower the direct treatment and direct spillover effects, so in equilibrium a tax decreases the average output of treated firms and increases the average output of untreated firms.

If $\beta_i < 0$ for all i, then firm i's output is a strategic complement with rivals' output. Since the direct treatment effect of a tax on any firm is strictly negative, the direct spillover effect of any taxed rival is a decrease in own output. In this case we need a stronger restriction to ensure MATTS is strictly negative: marginal profit is affected by a change in own output more than $n-1$ times as much as a change in any rival's output.

Interestingly, the restrictions on marginal profit are by firm—they are not global conditions. So it is possible to have $\beta_i < 0$ for some i and $\beta_i > 0$ for other i while maintaining MATTS< 0 under the same conditions. This suggests that the right way to think about spillovers in this context is how a firm's output decision is affected by, rather than affects, the output of rivals.

Finally, note that the homogeneous goods Cournot oligopoly arises when $\beta_i = \beta > 0$ and $b_{ii} = 2\beta$. But our framework accommodates a more general model as it allows each firm to face a different price and allows direct spillovers to vary in intensity and sign by firm.

8.1.3 Strategic Complements

If $b_{ij} \leq 0$ for all $j \neq i$ and $i = 1, ..., n$, then outputs are complements in demand and are strategic complements. By Theorem 4, MATTS is strictly negative and ASU is positive whenever the unit tax on any nontrivial subset of firms is increased if, for $i = 1, ..., n$,

$$b_{ii} + c_i''(\bar{y}_i) > (n-1) \max_{j \neq i} \{-b_{ij}\}. \tag{8.2}$$

Interpreted as marginal profit, this condition says that an increase in own output has at least $n-1$ times the impact in magnitude on marginal profit as a unit increase in any rival's output. Then ATTS is strictly negative, total output decreases, and, if only one firm is taxed, then the taxed firm decreases its output by more than the change in output of any rival.

8.1.4 Strategic Substitutes

If $b_{ij} > 0$ for all $j \neq i$ and $i = 1, ..., n$, then outputs are substitutes in demand and are strategic substitutes. By Theorem 5, if $n = 2$ or if $\frac{b_{ij}}{b_{ii}+c_i''(\bar{y}_i)} = \beta < 1$ for all i and $j \neq i$, then MATTS and ATTS are strictly negative while ASU is positive whenever the tax is increased on any nontrivial subset of firms. Note that the latter is a special case of spillovers that are anonymous-by-unit.

If $n \geq 3$, to reach these conclusions in general we require the normalized slope coefficients are not too heterogeneous. Specifically, if $0 < y \leq {}^{b_{ij}}/_{(b_{ii}+c_i(\bar{y}_i))} \leq x < 1$, we need $s \leq \frac{1}{n-2}$ where s satisfies $x^2 = sy + (1-s)y^2$.

8.1.5 Small Spillovers

Finally, by Theorem 6, in general we can say that MATTS and ATTS are strictly negative, total output decreases, and the output of a single treated firm ($n_t = 1$) decreases by more that the output of any other firm for any group assignment vector $g \in \Gamma$ as long as, for $i = 1, ..., n$,

$$b_{ii} + c_i''(\bar{y}_i) > (n-1)^2 \sum_{j \neq i} |b_{ij}| \,. \tag{8.3}$$

This is the only case considered in which different rivals can have differently signed direct spillover effects on a firm's marginal profit. But this sign heterogeneity comes at a cost. Condition (8.3) is a much stronger than condition (8.2), for example.

8.1.6 Generalization

While this analysis assumed linear demand and convex costs, it generalizes easily. Specifically, the negated Jacobian is the matrix of cross-partials $-\frac{\partial^2 \pi_i}{\partial y_i \partial y_j}$. The results obtain if in each of the conditions we replace b_{ij} with $-\frac{\partial^2 \pi_i}{\partial y_i \partial y_j}$ for the off-diagonal terms ($i \neq j$) and replace the diagonal terms, $b_{ii} + c_i''(\bar{y}_i)$, with $-\frac{\partial^2 \pi_i}{\partial y_i^2}$.

8.2 Perfect Competition

We now consider the case where firms are price-takers and produce a homogeneous good. Firms select output to maximize profit taking price as given,

$$\max_{y_i} py_i - c_i(y_i) - \gamma_i \lambda_i y_i.$$

Assume $c_i(y_i)$ is strictly increasing, strictly convex, and that $c_i''(0) = 0$. The profit-maximizing quantities satisfy

$$p - c_i'(y_i) - \gamma_i \lambda_i = 0 \text{ for } i = 1, ..., n.$$

Let inverse market demand be $p = D(\sum_{i=1}^n y_i)$ for $D : \mathbb{R} \to \mathbb{R}$ a strictly decreasing and differentiable function. Substitute this in to write

$$D(Y) - c_i'(y_i) - \gamma_i \lambda_i = 0 \text{ for } i = 1, ..., n.$$

Equilibrium outputs are a solution to this system. Its negated Jacobian is

$$\begin{bmatrix} -D'(\bar{Y}) + c_1''(\bar{y}_1) & -D'(\bar{Y}) & \cdots & -D'(\bar{Y}) \\ -D'(\bar{Y}) & -D'(\bar{Y}) + c_2''(\bar{y}_2) & \cdots & -D'(\bar{Y}) \\ \vdots & \vdots & \ddots & \vdots \\ -D'(\bar{Y}) & -D'(\bar{Y}) & \cdots & -D'(\bar{Y}) + c_n''(\bar{y}_n) \end{bmatrix}.$$

Spillovers are anonymous-by-unit and the negated Jacobian is a B-matrix since $c_i''(\bar{y}_i) > 0$ and $D'(\bar{Y}) < 0$. By Theorem 5, any increase in the tax on any nontrivial subset of firms implies that MATTS is strictly negative and total output decreases. A consequence of the last result is that market price increases with the tax. Moreover, the output of every treated firm (i.e., those whose tax increases) decreases and the output of every untreated firm increases. In this way, perfect competition is the ideal setting in which to test the profit maximization hypothesis via the comparative statics of taxation.

9 Conclusion

This main goal of this paper is to expand the empirical testability of theories within the DiD framework. This is important because economics as a discipline has generated a wealth of models and insights to explain human and market behavior. Researchers need simple methods to select the appropriate model to apply to their research question. We hope that this paper has provided a step in that direction.

We accomplished this task in two steps, each of which is of independent interest. First, we showed that within the canonical DiD framework, the population DiD identifies MATTS when spillovers, or interference, are present. MATTS reduces to ATT when SUTVA is maintained. This means that canonical DiD research designs that would normally be unacceptable exclusively because of spillovers can be used if the estimate is interpreted as MATTS.

It is important to get this interpretation right so that treatment effects can be appropriately assessed. We have illustrated that the classic DiD estimator exhibits a sign reversal property if interpreted as a causal estimate of ATT rather than MATTS—a researcher may conclude that a treatment has an adverse impact when then true impact is positive.

Second, within the context of a widely applicable metamodel with spillovers we derived novel conditions under which we can predict the sign of MATTS. These predictions are valid for any treatment group, so a statistically significant DiD estimate whose sign disagrees with the predicted sign constitutes a rejection of the theory.[15] Thus, a clear, straight line has been drawn in this paper between theoretical predictions and their reduced form empirical test using conventional comparative statics analysis.

In addition to its connection to empirical hypothesis testing, we have shown throughout that the comparative statics analysis yields new insights to old questions while also posing and answering new questions. These insights were facilitated by the observation that B-matrices play an important role in signing comparative statics. Developing this connection required us to derive a few new results for the class of B-matrices, especially with respect to conditions on the elements of a matrix under which its inverse is a B-matrix by columns.

[15] We avoid the term *refutation* and opt instead for *rejection* since a type I error cannot be ruled out.

A Appendix

Proof of Theorem 1

Proof. (1) For a given $g \in \Gamma$,

$$ATTS = \frac{1}{n_t} \sum_{r:g_r=1} \left(\sum_{s:g_s=1} \delta_{rs} f_\lambda^s \right) = \frac{1}{n_t} \sum_{s:g_s=1} \left(\sum_{r:g_r=1} \delta_{rs} \right) f_\lambda^s \geq 0$$

whenever $f_\lambda^s > 0$ for all $s : g_s = 1$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all $s : g_s = 1$. Since the result is for any $g \in \Gamma$, it follows that $ATTS \geq 0$ whenever $D_\lambda f > 0$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all $s : g_s = 1$ and all $g \in \Gamma$.

In addition, it is easy to see that, for a given $g \in \Gamma$, $ATTS > 0$ whenever $f_\lambda^s > 0$ for all $s : g_s = 1$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all $s : g_s = 1$ with strict inequality for at least one $s : g_s = 1$. The result follows since this must be true for any $g \in \Gamma$.

(2) Similarly, for a given $g \in \Gamma$ such that $n_t < n$,

$$ASU = \frac{1}{n_u} \sum_{r:g_r=0} \left(\sum_{s:g_s=1} \delta_{rs} f_\lambda^s \right) = \frac{1}{n_u} \sum_{s:g_s=1} \left(\sum_{r:g_r=0} \delta_{rs} \right) f_\lambda^s \geq 0$$

whenever $f_\lambda^s > 0$ for all $s : g_s = 1$ iff $\sum_{r:g_r=0} \delta_{rs} > 0$ for all $s : g_s = 1$. If $n_t = n$, then $ASU = 0$ by assumption. Since the result is for any $g \in \Gamma$, it follows that $ASU \geq 0$ whenever $D_\lambda f > 0$ iff $\sum_{r:g_r=0} \delta_{rs} \geq 0$ for all $s : g_s = 1$ and all $g \in \Gamma$.

If $n_t < n$, then $ASU > 0$ whenever $f_\lambda^s > 0$ for all $s : g_s = 1$ iff $\sum_{r:g_r=0} \delta_{rs} > 0$ for all $s : g_s = 1$,with strict inequality for at least one $s : g_s = 0$. The result follows since this must be true for any $g \in \Gamma$.

(3) Finally, for a given $g \in \Gamma$ such that $n_t < n$,

$$\begin{aligned} MATTS &= \frac{1}{n_t} \sum_{s:g_s=1} \left(\sum_{r:g_r=1} \delta_{rs} \right) f_\lambda^s - \frac{1}{n_u} \sum_{s:g_s=1} \left(\sum_{r:g_r=0} \delta_{rs} \right) f_\lambda^s \\ &= \sum_{s:g_s=1} \left(\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} - \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs} \right) f_\lambda^s \\ &\geq 0 \end{aligned}$$

whenever $f_\lambda^s > 0$ for $s : g_s = 1$ iff $\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \geq \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}$ for all $s : g_s = 1$. If $n_t = n$ so that $g = (1, 1, ..., 1)$,

$$MATTS = \frac{1}{n} \sum_{s=1}^{n} \left(\sum_{r=1}^{n} \delta_{rs} \right) f_\lambda^s \geq 0$$

if and only if $\sum_{r=1}^{n} \delta_{rs} \geq 0$ for $s = 1, ..., n$.

Since the result is for any $g \in \Gamma$, it follows that $MATTS \geq 0$ whenever $D_\lambda f > 0$ iff $\sum_{r:g_r=1} \delta_{rs} \geq 0$ for all $s = 1, ..., n$. A similar argument proves the result which characterizes $MATTS > 0$. □

Proof of Theorem 2

Proof. To prove this result, we will show that

$$\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \geq \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs} \text{ for all } s : g_s = 1, \text{ any } g \in \Gamma \text{ with } 1 \leq n_t < n; \text{ and} \tag{A.1}$$

$$\frac{1}{n} \sum_{r=1}^{n} \delta_{rs} \geq 0 \text{ for } s = 1, ..., n \tag{A.2}$$

is equivalent to the definition of a B_0-matrix by columns:

$$\frac{1}{n} \sum_{r=1}^{n} \delta_{rs} \geq c_s^+ \text{ for } s = 1, ..., n. \tag{A.3}$$

The result for B-matrices is obtained by using strict rather than weak inequalities in the "$\Leftarrow$" direction.

($\Rightarrow$) Fix $g \in \Gamma$. If $n_t = n - 1$, then (A.1) implies, for each $s = 1, ..., n$,

$$\frac{1}{n-1} \sum_{r:g_r=1} \delta_{rs} \geq \delta_{ks} \text{ for } k : g_k = 0.$$

Since this must be true for any group assignment $g \in \Gamma$, then for each $s = 1, ..., n$,

$$\frac{1}{n-1} \sum_{r \neq k, r=1}^{n} \delta_{rs} \geq \delta_{ks} \text{ for } k \neq s, k = 1, ..., n$$

$$\sum_{r \neq k, r=1}^{n} \delta_{rs} \geq (n-1)\delta_{ks} \text{ for } k \neq s, k = 1, ..., n$$

$$\sum_{r=1}^{n} \delta_{rs} \geq n\delta_{ks} \text{ for } k \neq s, k = 1, ..., n.$$

Combining these $n - 1$ inequalities with (A.2) results in (A.3).

($\Leftarrow$) Now assume that (A.3) holds. Then (A.2) is directly implied.

To show that (A.1) is also implied, let $1 \leq n_t < n$, fix $g \in \Gamma$, and note that by (A.3) we have, for $s = 1, ..., n$,

$$\sum_{r=1}^{n} \delta_{rs} \geq n\delta_{ks} \text{ for } k \neq s, k = 1, ..., n$$
$$\sum_{r:g_r=1} \delta_{rs} \geq n\delta_{ks} - \sum_{r:g_r=0} \delta_{rs} \text{ for } k \neq s, k = 1, ..., n.$$

Now fix a column s where $s : g_s = 1$ and sum over the $n_u > 0$ inequalities with k such that $g_k = 0$:

$$n_u \sum_{r:g_r=1} \delta_{rs} \geq n \sum_{k:g_k=0} \delta_{ks} - n_u \sum_{r:g_r=0} \delta_{rs}$$
$$n_u \sum_{r:g_r=1} \delta_{rs} \geq n_t \sum_{r:g_r=0} \delta_{rs}$$
$$\frac{1}{n_t} \sum_{r:g_r=1} \delta_{rs} \geq \frac{1}{n_u} \sum_{r:g_r=0} \delta_{rs}.$$

Since this inequality holds for any $s : g_s = 1$ and any group assignment vector $g \in \Gamma$, it must hold for $s = 1, ..., n$, as desired. □

Proof of Lemma 1

Proof. We establish a few facts to facilitate the proof. From $-D_y f[-D_y f]^{-1} = I$ it follows that, for $i \neq j$ and $j = 1, ..., n$, $\sum_{m=1}^{n} -f_m^i \delta_{mj} = -\alpha_i \delta_{ij} - \beta_i \sum_{m \neq i} \delta_{mj} = 0$. The last equality implies

$$\delta_{ij} = -\frac{\beta_i}{\alpha_i} \sum_{m \neq i} \delta_{mj}. \tag{A.4}$$

Next, let e^j be a column vector with 1 in the j^{th} position and zeros elsewhere, while $\nVdash$ is a column vector of ones. Put $g = e^j$, $D_\lambda f(\bar{y}; \bar{\lambda}) = \nVdash$ and recall that $G = Ig$, so by the implicit function theorem,

$$D(\bar{y}) = -[D_y f(\bar{y}; \bar{\lambda})]^{-1} G D_\lambda f(\bar{y}; \bar{\lambda}) = e^j.$$

This says that $D(y)$ equals the j^{th} column of $-[D_y f(\bar{y}; \bar{\lambda})]^{-1}$. Since the i^{th} element of $D(y)$ is $\frac{dy_i}{d\lambda_j} = \delta_{ij}$, it follows from (7.2) that, for $j = 1, ..., n$,

$$\sum_{m=1}^{n} \delta_{mj} = -\frac{1}{\Gamma} \frac{1}{\alpha_j - \beta_j}. \tag{A.5}$$

Next, for $i \neq j$ and $j = 1, ..., n$, we have from (A.4) that

$$\sum_{m \neq i} \delta_{mj} = \sum_{m=1}^{n} \delta_{mj} - \delta_{ij} = \sum_{m=1}^{n} \delta_{mj} + \frac{\beta_i}{\alpha_i} \sum_{m \neq i} \delta_{mj}.$$

This implies for $i \neq j$ and $j = 1, ..., n$ that

$$\sum_{m \neq i} \delta_{mj} = \frac{\alpha_i}{\alpha_i - \beta_i} \sum_{m=1}^{n} \delta_{mj}, \tag{A.6}$$

Finally, note that

$$\begin{aligned} & \sum_{m} \delta_{mj} > n\delta_{ij} \\ \Leftrightarrow \quad & \sum_{m \neq i} \delta_{mj} > (n-1)\delta_{ij} \\ \Leftrightarrow \quad & \sum_{m \neq i} \delta_{mj} > -(n-1)\frac{\beta_i}{\alpha_i} \sum_{m \neq i} \delta_{mj}, \end{aligned} \tag{A.7}$$

where the last inequality follows from (A.4).

($\Rightarrow$) Suppose $-D_\lambda f$ is a B-matrix. Then $\alpha_i < 0$. Also, as was shown in the main text,

$$\sum_{m=1}^{n} \delta_{mj} = -\frac{1}{\Gamma} \frac{1}{\alpha_j - \beta_j} > 0.$$

Moreover, for $i \neq j$ and $j = 1, ..., n$,

$$\sum_{m \neq i} \delta_{mj} = \frac{\alpha_i}{\alpha_i - \beta_i} \sum_{m=1}^{n} \delta_{mj} > 0 \tag{A.8}$$

since $\alpha_i/(\alpha_i - \beta_i) > 0$ from the fact that A is a B-matrix. Then inequality (A.7) is equivalent to

$$\alpha_i < -(n-1)\beta_i,$$

which is implied since A is a B-matrix. Thus, $[-D_\lambda f]^{-1}$ is a B-matrix by columns.

($\Leftarrow$) Now suppose $[-D_y f]^{-1}$ is a B-matrix by columns. We wish to show that

$$\begin{cases} \alpha_i < \beta_i & \text{if } \beta_i \leq 0 \\ \alpha_i < -(n-1)\beta_i & \text{if } \beta_i > 0. \end{cases}$$

To this end, note that

$$\sum_{m=1}^{n} \delta_{mj} = -\frac{1}{\Gamma}\frac{1}{\alpha_j - \beta_j} > 0 \text{ for } j = 1, ..., n$$

implies sgn $\left(-\frac{1}{\Gamma}\right)$=sgn $\left(\frac{1}{\alpha_j - \beta_j}\right)$ for $j = 1, ..., n$. It follows that either $\alpha_j - \beta_j > 0\ \forall j$ or $\alpha_j - \beta_j < 0\ \forall j$.

Suppose $\alpha_j - \beta_j < 0\ \forall j$. Since $[-D_y f]^{-1}$ is a B-matrix, for $j = 1, ..., n$,

$$\sum_{m=1}^{n} \delta_{mj} > n\delta_{ij} \text{ for all } i \neq j.$$

By subtracting δ_{ij} from both sides, substituting expression (A.4) for δ_{ij}, and substituting expression (A.8) for $\sum_{m\neq i} \delta_{mj}$, this is equivalent to

$$\begin{aligned} \sum_{m\neq i} \delta_{mj} &> (n-1)\delta_{ij} \text{ for all } i \neq j \\ \sum_{m\neq i} \delta_{mj} &> -(n-1)\frac{\beta_i}{\alpha_i}\sum_{m\neq i}\delta_{mj} \\ \frac{\alpha_i}{\alpha_i - \beta_i}\sum_{m=1}^{n} \delta_{mj} &> -(n-1)\frac{\beta_i}{\alpha_i - \beta_i}\sum_{m=1}^{n}\delta_{mj}. \end{aligned}$$

Divide both sides by $\sum_{m=1}^{n} \delta_{mj}/(\alpha_i - \beta_i)$ to get

$$\alpha_i < -(n-1)\beta_i.$$

Hence, $-D_y f$ is a B-matrix.

Now suppose $\alpha_j - \beta_j > 0\ \forall j$. The diagonal terms of the inverse of a B-matrix are positive (Christensen, 2019), so we must have $\alpha_j < 0$ since $\alpha_j \neq 0$ by assumption. It follows that $\beta_j < 0$ and (A.8) implies $\sum_{m\neq i} \delta_{mj} < 0$. Thus, (A.7) is equivalent to

$$\alpha_i > -(n-1)\beta_i > 0$$

which contradicts the the fact that $\alpha_i < 0$. Thus, we must have $\alpha_j - \beta_j < 0\ \forall j$. □

Proof of Theorem 4

Proof. (1) By equation (4.2), $-D_y f(\bar{y}; \bar{\lambda})D(\bar{y}) = GD_\lambda f(\bar{y}; \bar{\lambda})$. Since $-D_y f(\bar{y}; \bar{\lambda})$ is a Z-matrix and $GD_\lambda f \geq 0$, it follows that $D(\bar{y}) \geq 0$ if and only if $-D_y f(\bar{y}; \bar{\lambda})$ is an M-matrix

(Plemmons, 1977). It follows immediately that ATTS and ASU are positive.

(2) By Theorem 2 and Corollary 1 we need to show that the inverse of $-D_y f(\bar{y};\bar{\lambda})$ is a B-matrix by columns, or $\sum_j \delta_{ij} > nc_j^+$ for all $j = 1, ..., n$. To this end, notice that $-D_y f(\bar{y};\bar{\lambda})[-D_y f(\bar{y};\bar{\lambda})]^{-1} = I$ implies that for any $j = 1, ..., n$ and any $i \neq j$,

$$\begin{aligned}
-f_i^i \delta_{ij} &= \sum_{m\neq j} f_m^j \delta_{mj} \\
-f_i^i \delta_{ij} &\leq \max_{m\neq j}\{f_m^j\} \sum_{m\neq j} \delta_{mj} \\
\delta_{ij} &\leq \frac{1}{-f_i^i} \max_{m\neq j}\{f_m^j\} \sum_{m\neq j} \delta_{mj} \\
(n-1)\delta_{ij} &\leq \frac{n-1}{-f_i^i} \max_{m\neq j}\{f_m^j\} \sum_{m\neq j} \delta_{mj}.
\end{aligned}$$

Since $-f_i^i > (n-1)\max_{j\neq i}\{f_j^i\}$ we have

$$\frac{n-1}{-f_i^i} \max_{m\neq j}\{f_m^j\} < 1.$$

Hence, since $-D_y f(\bar{y};\bar{\lambda})$ is an M-matrix,

$$\begin{aligned}
0 \leq (n-1)\delta_{ij} &< \sum_{m\neq j} \delta_{mj} \\
0 \leq n\delta_{ij} &< \sum_{m=1}^{n} \delta_{mj},
\end{aligned}$$

as desired. □

Proof of Lemma 2

Proof. Let H be the diagonal matrix with elements $-1/f_i^i$ on the main diagonal. Then $-HD_y f(\bar{y};\bar{\lambda})$ is a strictly positive matrix with a unit diagonal. It follows from Willoughby (1977) that $-[HD_y f(\bar{y};\bar{\lambda})]^{-1} = -D_y f(\bar{y};\bar{\lambda})^{-1}H^{-1}$ is a SDD (by rows and columns) M-matrix.

H^{-1} is a diagonal matrix with diagonal elements $-f_i^i > 0$, so $-D_y f(\bar{y};\bar{\lambda})^{-1}$ has the same sign pattern as $-D_y f(\bar{y};\bar{\lambda})^{-1}H^{-1}$. Moreover, the inverse of $-D_y f(\bar{y};\bar{\lambda})$ is positive so $-D_y f(\bar{y};\bar{\lambda})$ must be an M-matrix.

Recall that δ_{ij} is the typical element of $[-D_y f(\bar{y};\bar{\lambda})]^{-1}$. Since $-D_y f(\bar{y};\bar{\lambda})^{-1}H^{-1}$ is SDD by columns, we have for $j = 1, ..., n$,

$$|\delta_{jj}| \left(-f_j^j\right) = \left|\delta_{jj} f_j^j\right| > \sum_{i \neq j} \left|\delta_{ij} f_j^j\right| = \left(-f_j^j\right) \sum_{i \neq j} |\delta_{ij}|.$$

Hence, $|\delta_{jj}| > \sum_{i \neq j,i}^{n} |\delta_{ij}|$, which proves that $[-D_y f(\bar{y}; \bar{\lambda})]^{-1}$ is SDD by columns. A similar argument shows that $[-D_y f(\bar{y}; \bar{\lambda})]^{-1}$ is SDD by rows. □

Proof of Theorem 5

Proof. Lemma 2 implies $[-D_y f]^{-1}$ is SDD by columns. By Fact 1 this implies $[-D_y f]^{-1}$ is a B-matrix by columns. Then by Theorem 2, MATTS> 0 for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$. The results of Corollary 1 therefore apply. Finally, the fact that $-D_y f(\bar{y}; \bar{\lambda})^{-1}$ is an M-matrix means that its off-diagonal terms are negative. Thus, Theorem 1 implies $ASU \leq 0$ for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$. In fact, for all i such that $g_i = 0$, $\frac{d\bar{y}_i}{d\lambda} = \sum_{j:g_j=1} \delta_{ij} f_\lambda^j \leq 0$ since $\delta_{ij} \leq 0$ for $i \neq j$ and $f_\lambda^j > 0$ for all j. □

Proof of Lemma 3

Proof. The fact that A is a SDD matrix is immediate from the definition. To see that A is also a B-matrix, observe that

$$a_{ii} + \sum_{j \neq i} a_{ij} > (n-1) \sum_{j \neq i} |a_{ij}| + \sum_{j \neq i} a_{ij} = \sum_{j \neq i} ((n-1)|a_{ij}| + a_{ij}) \geq 0$$

since each term in the last sum is positive. Moreover, for every $k \neq i$,

$$\begin{aligned} a_{ii} + \sum_{j \neq i} a_{ij} - n a_{ik} &> \sum_{j \neq i} ((n-1)|a_{ij}| + a_{ij}) - n a_{ik} \\ &= (n-1)(|a_{ik}| - a_{ik}) + \sum_{j \neq i,k} ((n-1)|a_{ij}| + a_{ij}) \qquad \text{(A.9)} \\ &\geq 0. \qquad \text{(A.10)} \end{aligned}$$

It follows that , for $i = 1, ..., n$, $a_{ii} > n r_i^+$, as desired. □

Proof of Theorem 6

Proof. Clearly, $-D_y f$ is SDD. It is thus invertible and from Ostrowski (1952) we know $\delta_{jj} > |\delta_{ij}|$ for $i \neq j, j = 1, ..., n$. Now $-D_y f[-D_y f]^{-1} = I$ implies $\sum_{m=1}^{n} -f_m^i \delta_{mj} = 0$ for

$i \neq j$, so we can write, for $i \neq j$, $i = 1, ..., n$,

$$
\begin{aligned}
-f_i^i \delta_{ij} &= \sum_{m \neq i} f_m^i \delta_{mj} \\
\left| f_i^i \delta_{ij} \right| &= \left| \sum_{m \neq i} f_m^i \delta_{mj} \right| \\
-f_i^i \left| \delta_{ij} \right| &\leq \sum_{m \neq i} \left| f_m^i \right| \left| \delta_{mj} \right| \\
-f_i^i \left| \delta_{ij} \right| &\leq \delta_{jj} \sum_{m \neq i} \left| f_m^i \right| \\
\left| \delta_{ij} \right| &\leq \delta_{jj} \frac{\sum_{m \neq i} \left| f_m^i \right|}{-f_i^i} \\
\left| \delta_{ij} \right| &< \delta_{jj} \frac{1}{(n-1)^2},
\end{aligned}
$$

where the last inequality follows by the assumption in the theorem statement.

Summing across all $n - 1$ inequalities for a given j we get

$$
\begin{aligned}
\sum_{i \neq j} \left| \delta_{ij} \right| &< \delta_{jj} \frac{1}{n-1} \\
(n-1) \sum_{i \neq j} \left| \delta_{ij} \right| &< \delta_{jj}.
\end{aligned}
$$

It follows from Lemma 3 that $[-D_y f]^{-1}$ is SDD by columns and a B-matrix by columns. Then Theorem 2 implies MATTS> 0 for any group assignment vector $g \in \Gamma$ whenever $D_\lambda f > 0$. □

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