Towson University
Department of Economics
**Working Paper Series**

Working Paper No. 2016-01

# Comparative Statics and Heterogeneity

by Finn Christensen

October, 2016

# Comparative Statics and Heterogeneity

Finn Christensen
Towson University

October 7, 2016

**Abstract**

This paper elucidates the role played by the heterogeneity of interactions between the endogenous variables of a model in determining the model's behavior. It is known that comparative statics are well-behaved if these interactions are relatively small, but the formal condition imposed on the Jacobian which typically captures this idea–diagonal dominance–ignores the distribution of the interaction terms. I provide a new condition on the Jacobian–*mean positive dominance*–which better captures a trade-off between the size and heterogeneity of interaction terms. In accord with Samuelson's (1947) correspondence principle, I also show that mean positive dominance yields stability and uniqueness results. Applications are provided to optimization problems, differentiable games, and competitive exchange economies.

*Keywords: comparative statics, heterogeneity, mean positive dominance, correspondence principle, B−matrix, stability, uniqueness, optimization, differentiable games, Cournot oligopoly, general equilibrium*

*JEL Classification: D11*

# 1 Introduction

Comparative statics analysis is complicated by interactions between endogenous variables. To see this, consider a game-theoretic context where the total equilibrium effect of a positive shock on a player's action can be decomposed into a partial effect and an interactions effect. The partial effect is the increase in the player's action holding constant all other players' actions. The interactions effect, which accounts for all the adjustments in players' best responses due to changes in other players' actions, is the difference between the total effect and partial effect. If there are negative interactions where a player's best response is decreasing in the action of some other player, then the partial effect and interactions effect may have opposite signs. Consequently, the total effect and partial effect may have opposite signs as well.

In broad terms, two situations give rise to comparative statics which are "well-behaved," meaning that the total effect has the same sign as the partial effect. The first is when the interactions effect has the same sign as the partial effect, and the second is when the interactions effect is small relative to the partial effect.

Monotone economic models tend to generate an interactions effect that has the same sign as the partial effect. This insight is well-known from the gross substitutes property in general equilibrium settings. More recently, without relying on purely mathematical assumptions associated with the implicit function theorem (IFT) like smoothness, the monotone comparative statics (MCS) literature has shown that the equilibrium set of a model with complementarity is increasing in a parameter if some form of the single crossing property between the endogenous variables and the parameter is satisfied.[1] However, many economic environments fall outside of the MCS framework or do not exhibit the requisite complementarity. With some notable exceptions[2] the lattice-based techniques developed in the MCS literature have yielded little insight in these settings. In fact, Roy and Sabarwal (2008) show that the set of equilibria in games with strategic substitutes is not a lattice in general.

To ensure that the interactions effect is small relative to the partial effect, in smooth models one typically places restrictions on the off-diagonal terms of the Jacobian relative to the terms along the main diagonal, for the off diagonal terms capture

[1] Early contributions to the monotone comparative statics literature include Topkis (1998), Vives (1990), Milgrom and Roberts (1990), and Milgrom and Shannon (1994).

[2] For example, Amir and Lambson (2000), Villas-Boas (1997), Acemoglu and Jensen (2013) and Hoerning (2003), among others.

interactions effect while the terms on the main diagonal contribute to the partial effect.[3] A standard way to formalize this idea is to require that the Jacobian is diagonally dominant, that is, in each row the absolute value of the term on the main diagonal is greater than the sum of the absolute values of the off-diagonal terms. But this condition alone is not enough for well-behaved comparative statics. For example, Dixit (1986) also requires that the off-diagonal terms are identical by rows, Simon (1989) requires monotonicity but allows for nonpositive shocks, and Jinji (2014) imposes an additional dominance condition on the minors of the Jacobian.

This paper captures another factor contributing to well-behaved comparative statics: homogeneity of interaction effects. Ill-behaved comparative statics can arise if strong negative interactions are concentrated among a subset of the equilibrium equations. But if the same cumulative interactions effect is distributed evenly across all equations, comparative statics may be well-behaved. Thus, a trade-off exists between the heterogeneity and magnitude of interaction effects in the sense that a larger cumulative magnitude can be tolerated if there is less heterogeneity, and vice versa, while still retaining well-behaved comparative statics. Diagonal dominance restricts only the cumulative magnitude, so in cases where there is homogeneity, this condition can place a stronger restriction on the magnitude of the interaction effects than is required. In fact, I show that the new condition I provide generalizes the results in Dixit (1986) since homogeneity is imposed in an *ad hoc* manner in that paper.

Formally, I require that, for each row (column) of the Jacobian, the mean is positive and larger than each of the off-diagonal elements. This *mean positive dominance* property is simple to verify and is sufficient for well-behaved comparative statics at the aggregate (individual) level for certain parameter shocks. The result for the aggregate, defined as an increasing function of the sum of the equilibrium vector's elements, applies to any positive partial effect. The individual level result applies to the $i$th element of the equilibrium vector for shocks which are *mean positive dominant in element* $i$, that is, where the average partial effect is positive and greater than each of the partial effects to elements other than $i$. Existing individual level results in non-monotone models typically restrict attention to shocks that are positive for the *ith* element and zero elsewhere (e.g., Dixit 1986; Corchón, 1994; Acemoglu and Jensen, 2013). These idiosyncratic shocks are a subset of mean positive dominant

[3] See Roy and Sabarwal (2010) and Monaco and Sabarwal (2015) for applications of this insight to nonsmooth environments.

shocks.

In accord with Samuelson's (1947, especially Chapter 9) "correspondence principle," a principle which refers to the intimate connection between comparative statics and stability, I also show that equilibrium is stable if the mean positive dominance property is satisfied subject to some additional conditions.[4] In addition, if the mean positive dominance condition applies globally (or only locally but with additional restrictions) then there is at most one equilibrium.

These powerful conclusions build off of fairly recent results from the linear algebra literature. Carnicer, Goodman and Peña (1999) show that matrices with the mean positive dominance property, termed $B-$matrices in Peña (2001), have a strictly positive determinant. The class of $B-$matrices appears to be new to the economics literature. These matrices have a rich set of properties including: $B-$matrices are $P-$matrices; a symmetric $B-$matrix is positive definite; a $B-$matrix whose transpose is also a $B-$matrix is positive stable and positive definite; and if the sum of a matrix and its transpose is a $B-$matrix then this sum and each component of the sum is positive definite.

These properties are often imposed on the Jacobian of an equilibrium system in order to prove "nice" properties of smooth economic models. For example, the $P-$matrix property allows one to apply Gale and Nikaido's (1965) global univalence result to prove uniqueness. Rosen's (1965) "diagonal strict concavity" condition for uniqueness in concave games is satisfied if the sum of the Jacobian of implicit best response functions and its transpose is negative definite. The index-theory-based uniqueness theorems in Varian (1975), Kehoe (1980) and Kolstad and Mathiesen (1987) require a positive determinant. Matrix stability is important for stability in continuous time, and so on. However, these mathematical conditions have no inherent economic interpretation, so it is useful to find meaningful conditions under which these properties are satisfied. The class of $B-$matrices provides some progress on that front.

Several applications in Section 4 enable comparison with the literature and illustrate the broad reach of the results. In the first application I apply the results to the

[4] Although Samuelson (1947) coined the term, he never provided a formal definition, and therefore different authors define the correspondence principle differently. However, Samuelson did illustrate this principle through a series of examples, and it is clear from these examples that he thought of the correspondence principle as referring to a large intersection between the set of conditions which imply well-behaved comparative statics and the set of conditions which imply stable dynamics.

comparative statics of optimization problems. This task is facilitated by the fact that a symmetric $B-$matrix is positive definite. Consequently, if the Hessian of a function is a $B-$matrix, then it is strictly convex. I call such functions $B-convex$. Equipped with this observation, I show that if the objective function of an unconstrained optimization problem is $B-$concave at a solution, then the sum of the solution vector is increasing in the parameter if the objective function has increasing differences, and that an element of the solution vector is increasing if the shock is mean positive dominant in the same element. This result is then applied to generalize some comparative statics conclusions in Rochet and Tirole (2003) regarding platform monopolies.

The results can also be applied to the comparative statics of constrained optimization problems such as the consumer's utility maximization problem. To this end, I provide a new condition on the utility function under which all goods are normal. The condition is that the average percentage change in marginal utility of all goods caused by a unit increase in one good $i$ is nonpositive and no greater than the percentage change in the marginal utility of any particular good different from $i$. In contrast to Chipman (1977) and Quah (2007) who require all goods to be complements, this condition allows some (or all) goods to be substitutes.

The next application is to games with convex strategy sets and differentiable payoff functions. When applied to the standard Cournot oligopoly model, the comparative statics conclusions reduce to the union of conditions provided in Dixit (1986) and Corchón (1994). In this way, the conditions provided can be viewed as a generalization of the Dixit-Corhón conditions to a very general differentiated products Cournot oligopoly. In fact, the case where each firm has its own inverse demand function and costs depend on other firms' output is included in this category.

I turn to competitive exchange economies in the final application. In this context, it is well-known that gross substitution imparts nice uniqueness, comparative statics, and stability properties on the economy. It turns out that mean positive dominance in partial price effects delivers similar properties yet allows some goods to be complements. Mean positive dominance formalizes the idea that the general equilibrium model is well-behaved when own-partial price effects dominate cumulative cross partial price effects. Also, since a change in the endowment of one good can change the consumption of all other goods through wealth effects, this application depends crucially on the fact that the main comparative statics result allows for parameter shocks which are mean positive dominant.

The next section establishes the general framework of analysis. In Section 3 I define and establish properties of $B-$matrices and use these properties to prove comparative statics, stability and uniqueness results. A subsection is dedicated to discussing how mean positive dominance captures the heterogeneity-magnitude trade-off. Section 4 contains applications and Section 5 concludes.

# 2 The Environment

Consider the function $f : X \times T \rightarrow \mathbb{R}^n$ where $X \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^s$ are open sets. The component functions of $f$ are $f_i : X \times T \rightarrow \mathbb{R}$ for $i = 1, ..., n$. For the sake of clarity and interpretation, we assume $s = 1$, but the results generalize to $s$ finite.

Given a parameter $\bar{t} \in T$, call $\bar{x} = x(\bar{t}) \in X$ an equilibrium vector if

$$f(\bar{x}; \bar{t}) = 0. \tag{1}$$

Note that if $f(x; \bar{t}) = x - g(x; \bar{t})$, then $\bar{x}$ is a fixed point of $g$. Since our interest is in comparative statics, assume directly that an equilibrium exists.

Also, restrict attention to equilibria at which $f$ is continuously differentiable and $\det D_x f(\bar{x}; \bar{t}) \neq 0$, where $D_x f(\bar{x}; \bar{t})$ is the Jacobian of $f$ evaluated at $(\bar{x}, \bar{t})$. By the IFT, the effect of a marginal increase in the parameter $t$ on the equilibrium vector $x(\bar{t})$ is

$$Dx(\bar{t}) = -[D_x f(\bar{x}; \bar{t})]^{-1} D_t f(\bar{x}; \bar{t}). \tag{2}$$

More insight into comparative statics can be gained if we assume $\frac{\partial f_i(\bar{x};\bar{t})}{\partial x_i} \neq 0$ for all $i$. Then letting $f_{ij} \equiv \frac{\partial f_i(x;t)}{\partial x_j}$ and totally differentiating system (1) at $(\bar{x}, \bar{t})$ we get

$$\underbrace{\frac{dx_i(\bar{t})}{dt}}_{TE} = \underbrace{-\frac{\partial f_i}{\partial t}\frac{1}{f_{ii}}}_{PE} + \underbrace{\sum_{j \neq i, j=1}^{n} -\frac{f_{ij}}{f_{ii}}\frac{dx_j(\bar{t})}{dt}}_{IE}, \; i = 1, ..., n. \tag{3}$$

The *total effect* $(TE)$ of a parameter change on $\bar{x}_i$ can be decomposed into a *partial effect* $(PE)$ and an *interactions effect* $(IE)$. The *interaction terms*, $(-f_{ij}/f_{ii})_{j \neq i}$ describe how the value of $\bar{x}_i$ changes in response to a one unit increase in $\bar{x}_j$, and the interactions effect on $\bar{x}_i$ aggregates the interaction terms scaled by the total effect on

$\bar{x}_j$.[5]

In matrix notation, let $\Lambda$ be a diagonal matrix with diagonal entries $(1/f_{11}, 1/f_{22}, ..., 1/f_{nn})$. Then the *matrix of interaction terms* is $I - \Lambda D_x f(\bar{x};\bar{t})$ and the *vector of partial effects* is $-\Lambda D_t f(\bar{x};\bar{t})$. The system of total effects (3) can be written

$$\begin{aligned} Dx(\bar{t}) &= -\Lambda D_t f(\bar{x};\bar{t}) + (I - \Lambda D_x f(\bar{x};\bar{t}))Dx(\bar{t}), \text{ or} \\ \Lambda D_x f(\bar{x};\bar{t})Dx(\bar{t}) &= -\Lambda D_t f(\bar{x};\bar{t}), \end{aligned}$$

which is equivalent to equation (2). I will refer to $\Lambda D_x f(\bar{x};\bar{t})$ as the *normalized Jacobian.*

Let $H : \mathbb{R} \to \mathbb{R}$ be a $C^1$ and increasing function, and let $H(\bar{\Sigma}) \equiv H(\Sigma_{i=1}^n x_i(\bar{t}))$ be the *equilibrium aggregate.* The main objective of this paper is to find sufficient conditions under which the equilibrium aggregate, or the *ith* element of the equilibrium vector, $\bar{x}_i$, is locally increasing in $t$ without actually inverting the Jacobian. The conditions I provide can be applied to $[D_x f(\bar{x};\bar{t})$ and $D_t f(\bar{x};\bar{t})]$ or to $[\Lambda D_x f(\bar{x};\bar{t})$ and $\Lambda D_t f(\bar{x};\bar{t})]$.

The equilibrium of many economic models can be analyzed in this framework, but the following model will be used to help with the exposition of the results.

**Demand for social goods.** Suppose $n \geq 2$ consumers indexed by $i$ allocate income $m_i$ between good $\mathcal{X}$ with price $p_x$ and good $\mathcal{Y}$ with price $p_y$. Consumer $i's$ preferences are represented by the $C^1$, strictly quasiconcave utility function $u_i(x_i, y_i, x_{-i})$, where $x_i$ and $y_i$ are consumer $i's$ consumption levels of goods $\mathcal{X}$ and $\mathcal{Y}$, while $x_{-i} = \{x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\}$ is the vector of others' consumption of good $\mathcal{X}$. In this sense good $\mathcal{X}$ is a *social good* while good $\mathcal{Y}$ is a *private good.*[6]

Consumers solve $\max_{(x_i,y_i)\in\mathcal{B}_i} u_i(x_i, y_i, x_{-i})$, where $\mathcal{B}_i = \{(x_i, y_i) \geq 0 : p_x x_i + p_y y_i \leq m_i\}$ is the set of affordable consumption bundles. If demand can be solved explicitly, denote the unique solution to this problem as

$$\bar{x}_i = g_i(x_{-i}; p, m_i) \text{ and } \bar{y}_i = h_i(x_{-i}; p, m_i),$$

[5] Applied to a game theoretic context where each equation represents a player's best response function, there is a clear connection to the social interactions literature. In this setting, say that the interactions effect *reinforces* the partial (or private) effect if $sgn(PE) = sgn(IE) \neq 0$; the interactions effect *counteracts* the private effect if $sgn(PE) = -sgn(IE) \neq 0$. The *social multiplier* is $\frac{TE}{PE}$, and this is greater than one if and only if the interactions effect has the same sign as the partial effect.

[6] To use alternative terminology, this is a model of interdependent preferences (e.g., Pollak, 1976)

where $g_i$ and $h_i$ are consumer $i's$ demand functions for goods $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Focus on the market for the social good. Only pure strategy equilibria exist since $\bar{x}_i$ is unique. Letting $m = (m_1, ..., m_n)$, an equilibrium demand system given $(p, m)$ is defined as

$$f(\bar{x}; \bar{p}, \bar{m}) \equiv \bar{x} - g(\bar{x}; \bar{p}, \bar{m}) = 0 \tag{4}$$

Since every individual's demand is continuous and constrained to $\mathcal{B}_i$, $g$ is continuous and maps a compact and convex set into itself. Therefore, an equilibrium exists by Brouwer's fixed point theorem.

In this setting the vector of parameters is $t = (p, m)$, but of primary interest is how the market demand for the social good (i.e., the equilibrium aggregate), $\bar{\Sigma}(p_x) = \Sigma_{i=1}^{n} g_i(\bar{x}_i; \bar{p}_x)$, varies with price $p_x$ when demand is differentiable at equilibrium and good $\mathcal{X}$ is not a Giffen good: $\frac{\partial g_i}{\partial p_x} \leq 0$ for all $i$.

# 3 Comparative Statics and Mean Positive Dominance

Consider the linear system of equations

$$Ay = b \tag{5}$$

where $A$ is an $n \times n$ matrix of real coefficients, $y$ is an $n \times 1$ vector of real variables, and $b$ is an $n \times 1$ vector of real parameters. If $A$ is invertible then $y = A^{-1}b$. This is the format of the comparative statics equation (2), so the main objective of this paper is formally equivalent to determining conditions on $A$ and $b$ under which $\sum_{i=1}^{n} y_i \geq 0$ and $y_i \geq 0$. The analysis will be conducted in this framework.

I begin with some elementary observations which highlight the importance of the row sums and column sums of $A^{-1}$. If $A$ is invertible, let $A^{-1} = \Delta$ be its inverse with typical element $\delta_{ij}$. Call $\sum_{i=1}^{n} \delta_{ij}$ the *jth inverse column sum of* $A$, and let $\mathcal{NICS}$ be the class of invertible matrices with nonnegative inverse column sums. Similarly call $\sum_{j=1}^{n} \delta_{ij}$ the *ith inverse row sum of* $A$, and let $\mathcal{NIRS}$ be the class of invertible matrices with nonnegative inverse row sums. The vector $b$ is *positive* if $b \geq 0$; the vector $b$ is *uniform* if $b_i = b_j$ for all $i, j \in \{1, ..., n\}$; the vector $b$ is *positive only in element* $i$ if $b_i > 0$ and $b_j = 0$ for all $j \neq i$.

The proof of the following lemma is relegated to the Appendix, as is any proof not contained in the text.

**Lemma 1** *Consider the linear system $Ay = b$ given in (5).*

1. *$\sum_{i=1}^{n} y_i \geq 0$ whenever $b$ is positive if and only if $A \in \mathcal{NICS}$.*

2. *$y_i \geq 0$ whenever $b$ is positive only in element $i$ if and only if $\delta_{ii} \geq 0$.*

3. *$y \geq 0$ whenever $b \neq 0$ is positive and uniform if and only if $A \in \mathcal{NIRS}$.*

Going forward, the theory of $B-$matrices plays a central role in the analysis. A *$B-$matrix* is a square matrix whose row means are positive and larger than each of the off-diagonal terms of the same row. Precisely, for each $i = 1, ..., n$, let

$$a_i^+ = \max\{0, a_{ij} | j \neq i\}.$$

Then the $n \times n$ matrix $A = (a_{ij})$ is a $B-$matrix if and only if, for all $i = 1, ..., n$,

$$\sum_{j=1}^{n} a_{ij} > n a_i^+. \tag{6}$$

In economic applications it will be helpful to allow for a weak version of $B-$matrices. Call $A$ a $B_0-$matrix if, for all $i = 1, ..., n$,

$$\sum_{j=1}^{n} a_{ij} \geq n a_i^+. \tag{7}$$

Clearly, every $B-$matrix is a $B_0-$matrix.

The term "$B-$matrix" is introduced in Peña (2001). A more descriptive moniker may be "mean positive dominant" matrices, and at times I will refer to inequalities (6) and (7) as the *(strict) mean positive dominance* condition.

The next lemma asserts that mean positive dominance is preserved under positive row scaling, addition of nonnegative numbers to the diagonal, and matrix addition. The last property is important for aggregation. The trivial proof is omitted.

**Lemma 2** *Let $A$ and $A'$ be $B(B_0)-$matrices. Let $D$ be a diagonal matrix with a strictly positive diagonal, and let $D'$ and $D''$ be nonnegative diagonal matrices. Then $DA + D'A' + D''$ is a $B(B_0)-$matrix.*

$B-$matrices and $B_0-$matrices possess many useful properties. Carnicer, Goodman, and Peña (1999) prove that a $B-$matrix has a strictly positive determinant. Peña (2001) demonstrates that $B-$matrices have strictly positive diagonals and that the principal submatrices of $B-$matrices are also $B-$matrices, which implies $B-$matrices are $P-$matrices.[7] Certainly, the same conclusions apply if the transpose of a matrix is a $B-$matrix since the determinant of a matrix equals the determinant of its transpose. The next lemma extends a weak version of these results to $B_0-$matrices and provides some new properties that will be used in this paper.

**Lemma 3** *Let $A$ or $A^T$ be a $B_0-$matrix. Then*

1. $\det A \geq 0$;
2. *the principal submatrices of $A$ are also $B_0-$matrices;*
3. *$A$ is a $P_0-$matrix*[8]*;*
4. *if $A$ is also symmetric, then $A$ is positive semidefinite;*
5. *$A$ has a nonnegative diagonal;*
6. *if $A$ and $A^T$ are $B(B_0)-$matrices, then $A+A^T$, $A$ and $A^T$ are positive (semi)definite*[9]*;*
7. *if $A+A^T$ is a $B(B_0)-$matrix, then $A+A^T$, $A$ and $A^T$ are positive (semi)definite; and*
8. *if $A$ and $A^T$ are $B-$matrices, then $A$ is positive stable.*[10]

The next lemma provides a tight relationship between the $B_0-$matrix property and the inverse row and inverse column sums. This result is central to the main comparative statics theorem, so a proof is provided here.

**Lemma 4** *1. If $A$ is an invertible $B_0-$matrix, then $A \in \mathcal{NICS}$ and $\delta_{ii} \geq 0$ for all $i$.*

[7] A $P-$matrix is a matrix with positive principal minors.

[8] A $P_0-$matrix is a matrix with nonnegative principal minors.

[9] $A$ is positive (semi)definite if $z'Az(\geq) > 0$ for every nonzero vector $z \in \mathbb{R}$. $A$ does not need to be symmetric.

[10] A matrix is *positive stable* if all of its eigenvalues have positive real parts.

2. *If $A^T$ is an invertible $B_0-$matrix, then $A \in \mathcal{NIRS}$ and $\delta_{ii} \geq 0$ for all $i$.*

**Proof.** (1) First suppose $A$ is an invertible $B_0-$matrix. Let $\Gamma = \left(\gamma_{ij}\right)$ be the cofactor matrix of $A$. Since $A^{-1} = \frac{\Gamma^T}{\det A}$, the inverse column sums are $\sum_{j=1}^n \frac{\gamma_{ij}}{\det A}$. Since $A$ is an invertible $B_0-$matrix we have $\det A > 0$, and by extension $\sum_{j=1}^n \gamma_{ij} \geq 0$ for all $i$. To see this, let

$$A^i(1) = \begin{bmatrix} a_{11} & a_{12} & \cdots & & \cdots & a_{1n} \\ a_{21} & a_{22} & & & & a_{2n} \\ \vdots & & \ddots & & & \vdots \\ 1 & 1 & & 1 & & 1 \\ \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & & \cdots & a_{nn} \end{bmatrix}$$

be the matrix obtained by replacing the $ith$ row of $A$ with ones. Clearly, $A^i(1)$ is a $B_0-$matrix so expanding along the $ith$ row we have $\det A^i(1) = \sum_{j=1}^n \gamma_{ij} \geq 0$ for all $i$. Thus, $A \in \mathcal{NICS}$.

To see that $A^{-1}$ has a nonnegative diagonal, note that the $(i,i)th$ element of $A^{-1}$ is $\delta_{ii} = \frac{\gamma_{ii}}{\det A}$, and $\gamma_{ii}$ is the $(i,i)$ minor of $A$. By parts 1 and 2 of Lemma 3, $\gamma_{ii} \geq 0$.

(2) Now suppose $A^T$ is an invertible $B_0-$matrix. Then, as just shown, $A^T \in \mathcal{NICS}$. It follows from $(A^T)^{-1} = (A^{-1})^T$ that $A \in \mathcal{NIRS}$. The proof that $A^{-1}$ has a nonnegative diagonal is the same for when $A$ is an invertible $B_0-$matrix. ■

We need a few more definitions before stating the main comparative statics result of this section. Define

$$b_{-i}^+ = \max\left\{b_1, ..., b_{i-1}, b_{i+1}, ..., b_n\right\}$$

as the largest element of $b$ excluding element $i$. Then the vector $b$ is *mean positive dominant in element* $i$ if the sum of the elements of $b$ is nonnegative and the value of every element but $i$ is less than the mean: $\sum_{i=1}^n b_i \geq \max\left\{0, nb_{-i}^+\right\}$. Say that the vector $b$ is *strictly mean positive dominant in element* $i$ if the inequality is strict: $\sum_{i=1}^n b_i > \max\left\{0, nb_{-i}^+\right\}$.

**Theorem 1 (Comparative Statics)** *Consider the linear system $Ay = b$.*

1. *Let $A$ be an invertible $B_0-$matrix. Then*

*(a)* $\sum_{i=1}^{n} y_i \geq 0$ *whenever* $b$ *is positive,*

*(b)* $y_i \geq 0$ *whenever* $b$ *is positive only in element* $i$*, and*

*(c) the inequality is reversed in parts (a) and (b) when* $-b$ *is positive and, respectively, when* $-b$ *is positive only in element* $i$*.*

*2. Let* $A^T$ *be an invertible* $B_0$*-matrix. Then*

*(a)* $y \geq 0$ *whenever* $b$ *is positive and uniform,*

*(b)* $y_i \geq 0$ *if* $b$ *is mean positive dominant in element* $i$*,*

*(c)* $y_i > 0$ *if* $A^T$ *is a* $B$*-matrix and* $b$ *is strictly mean positive dominant in element* $i$*, and*

*(d) the inequality is reversed in part (a) when* $-b$ *is positive and uniform, and in parts (b) and (c) if* $b = -c$ *and* $c$ *is (strictly) mean positive dominant.*

**Proof.** (1) and (2a) follow from Lemmas 1 and 4.

(2b) By Cramer's rule, $y_i = \frac{\det A_i}{\det A}$, where $A_i$ is the matrix obtained from $A$ by replacing column $i$ with the vector $b$. $\det A > 0$ since $A^T$ is a $B-$matrix and $\det A^T = \det A$. If $b$ is mean positive dominant in element $i$, then $A_i^T$ is a $B_0-$matrix. Hence, $\det A_i \geq 0$.

(2c) If $A^T$ is a $B-$matrix and $b$ is strictly mean positive dominant then $A_i^T$ is a $B-$matrix, so $\det A_i > 0$.

(2d) The first part is obvious. If $b = -c$ then $y_i = \frac{-\det \tilde{A}_i}{\det A}$, where $\tilde{A}_i$ is the matrix obtained from $A$ by replacing column $i$ with the vector $c$. ∎

When applied to the comparative statics equation (2), Theorem 1 implies that the equilibrium aggregate is locally nondecreasing in $t$, $dH(\bar{\Sigma})/dt \geq 0$, whenever

1. $D_x f(\bar{x}; \bar{t})$ is an invertible $B_0-$matrix and $-D_t f(\bar{x}; \bar{t}) \geq 0$, or

2. $-D_x f(\bar{x}; \bar{t})$ is an invertible $B_0-$matrix and $D_t f(\bar{x}; \bar{t}) \geq 0$, or

3. $\Lambda D_x f(\bar{x}; \bar{t})$ is an invertible $B_0-$matrix and $-\Lambda D_t f(\bar{x}; \bar{t}) \geq 0$.

Similarly, the $ith$ element of the equilibrium vector $\bar{x}$ is nondecreasing in $t$, $dx_i(\bar{t})/dt \geq 0$, if

1. $[D_x f(\bar{x};\bar{t})]^T$ is an invertible $B_0-$matrix and $-D_t f(\bar{x};\bar{t})$ is mean positive dominant in element $i$, or

2. $[-D_x f(\bar{x};\bar{t})]^T$ is an invertible $B_0-$matrix and $D_t f(\bar{x};\bar{t})$ is mean positive dominant in element $i$, or

3. $[\Lambda D_x f(\bar{x};\bar{t})]^T$ is an invertible $B_0-$matrix and $-\Lambda D_t f(\bar{x};\bar{t})$ is mean positive dominant in element $i$.

A few remarks are in order. First, a $B-$matrix must have a strictly positive diagonal while a $B_0-$matrix has a nonnegative diagonal. Since the normalized Jacobian has ones on the main diagonal, Theorem 1 cannot be applied to $-\Lambda D_x f(\bar{x};\bar{t})$. Second, if $\Lambda \geq 0$, then $D_x f(\bar{x};\bar{t})$ is a $B(B_0)$-matrix if and only if $\Lambda D_x f(\bar{x};\bar{t})$ is a $B(B_0)-$matrix by Lemma 2. These facts allow for some flexibility when verifying the mean positive dominance condition.

The question also arises as to the way forward with comparative statics if, in some model, it is natural that $f_{ii} < 0$ and $f_{jj} > 0$ for some $i \neq j$. Certainly, it is possible to apply Theorem 1 to the normalized Jacobian and vector of partial effects as long as $f_{kk} \neq 0$ for any $k \in \{1, ..., n\}$.[11] But sometimes it may be more convenient to deal directly with the (non-normalized) Jacobian. This is possible. For example, in the context of the linear system $Ay = b$, suppose $A = NM$ and $b = Nq$ where $M^T$ is a $B_0-$matrix, $q$ is mean positive dominant in element $i$, and $N$ is a diagonal matrix whose diagonal entries belong to the set $\{-1, 1\}$. Then $y_i \geq 0$ since $NMy = Nq$ iff $My = q$. The analogous technique can be applied to analyze aggregate comparative statics.

Theorem 1 is notable for several reasons. First, the terms of the Jacobian may take any sign, meaning that any type of interaction is allowed. This stands in contrast to existing comparative statics results which restrict the sign of interaction in some way. For example, Simon (1989) studies the case where $A$ is diagonally dominant with a strictly positive diagonal and nonpositive off-diagonals. Dixit (1986) studies diagonally dominant matrices whose off-diagonal terms are identical by rows. In justifying this assumption, Dixit writes (p. 119) that without it, "no structure could be imposed on [the matrix] inverse, and no meaningful results could emerge." Jinji (2014) allows for any sign, but the condition requires diagonal dominance as well

[11] If $f_{kk} = 0$ for some $k$ and the Jacobian or its negation is a $B_0-$matrix, then the $kth$ row contains only zeros, so the Jacobian is not invertible.

as a restriction on the minors of order $(n-2)$ of the Jacobian, so it is not clear how much insight is gained over simply inverting the Jacobian. Finally, the MCS literature restricts attention to non-negative off-diagonal terms, a point which will become clearer in the application to optimization problems below.

Second, apart from the literature on aggregative games, this is one of the few results of which I am aware that provides comparative statics results for the equilibrium aggregate which is not a simple corollary to monotonicity results. The comparative statics of the equilibrium aggregate has applications to contests, the slope of market demand for social goods, average relative price levels in general equilibrium, and industry output in Cournot oligopoly, among others.

Third, many existing comparative statics results for the *ith* element of $\bar{x}$ assume that the partial effect is positive only in element $i$ (e.g., Dixit 1986; Cochón, 1994; Acemoglu and Jensen, 2013).[12] Part 1b of Theorem 1 addresses this case directly, but parts 2b-c include it as a special case as partial effects that are positive only in element $i$ are mean positive dominant in element $i$. The generalization to mean positive dominant partial effects will prove particularly important in the application to general equilibrium.

Fourth, the result provides new insight into the forces which produce well-behaved comparative statics. Diagonal dominance formalizes the intuition that well-behaved comparative statics should be expected when the interaction terms are limited in *magnitude*. However, in a point which I will develop at length in the next subsection, mean positive dominance illustrates that the *heterogeneity* of interaction effects plays an important role, too.

Finally, $B-$matrices can also be used for stability, uniqueness and existence results. Uniqueness follows from Gale and Nikaido's (1965) classical result; both existence and uniqueness follow from index theory (e.g., Dierker (1972) and Varian (1975)).

For stability, in the tradition of general equilibrium tâtonnement it is common to assume that the dynamics of an economic model are governed by the differential equations

$$\frac{dx_i}{d\tau} = c_i f_i(x;\bar{t}) \text{ for } i = 1, ..., n, \tag{8}$$

where $\tau$ is the time variable and $c_i > 0$ is a positive constant. The idea of system

[12]One exception is Simon (1989), but the analysis is limited to the context described two paragraphs above.

(8) is that the speed of adjustment is positively related the distance away from an equilibrium, where an equilibrium $\bar{x}$ arises when $f(\bar{x};\bar{t}) = 0$. Refer to this model as *tâtonnement dynamics.*

Let $C$ be the diagonal matrix with diagonal entries $(c_1, ..., c_n)$. It is known that equilibrium $\bar{x}$ is locally asymptotically stable if and only if eigenvalues of the Jacobian of system (8), $CD_x f(\bar{x};\bar{t})$, have negative real parts. We then have the following result.

**Theorem 2 (Stability)** *Assume $f_{ii}(\bar{x};\bar{t}) < 0$ and consider tâtonnement dynamics described by system (8). Any of the following conditions is sufficient for $\bar{x}$ is locally asymptotically stable.*

1. $-D_x f(\bar{x};\bar{t})$ *is a symmetric $B-$matrix.*
2. $-D_x f(\bar{x};\bar{t})$ *and* $[-D_x f(\bar{x};\bar{t})]^T$ *are $B-$matrices and $c_i = \bar{c} > 0$ for all $i$.*

**Proof.** (1)-(2)The proof of each statement applies part 8 of Lemma 3, which requires us to show only that $-CD_x f(\bar{x};\bar{t})$ and $[-CD_x f(\bar{x};\bar{t})]^T$ are $B-$matrices. That $-CD_x f(\bar{x};\bar{t})$ is a $B-$matrix is immediate from Lemma 2 since $C$ is a diagonal matrix with a positive main diagonal. Since $[-CD_x f(\bar{x};\bar{t})]^T = [-D_x f(\bar{x};\bar{t})]^T C^T$, it follows that $[-CD_x f(\bar{x};\bar{t})]$ is a $B-$matrix if $-D_x f(\bar{x};\bar{t})$ is symmetric, or if $[-D_x f(\bar{x};\bar{t})]^T$ is also a $B-$matrix and $c_i = \bar{c}$ is constant for all $i$. ■

**Theorem 3 (Uniqueness)** *Consider the system of equations (1) and fix $\bar{t} \in T$.*

1. *Let $X$ be an open (closed) rectangle. If $D_x f(x;\bar{t})$ or $[D_x f(x;\bar{t})]^T$ is an invertible $B_0(B)-$matrix for all $x \in X$, then there is at most one equilibrium $\bar{x}$.*
2. *Let $M \subset \mathbb{R}^n$ be the closed $n-$dimensional unit disk.*[13] *Let $X$ be an open set containing $M$. Let $f : X \times T \to \mathbb{R}^n$ be smooth and such that for each $\bar{x} \in \mathcal{E}$, where*
$$\mathcal{E} = \{\bar{x} \in M | f(\bar{x};\bar{t}) = 0\}$$
*is the set of equilibrium points in $M$, $D_x f(\bar{x};\bar{t})$ is an invertible $B_0-$matrix. Assume $f$ points outward on the boundary of $M$. Then there is exactly one equilibrium.*

[13]This disk is defined as $D^n = \left\{x \in \mathbb{R}^n | \sum_i^n x_i^2 \leq 1\right\}$.

**Proof.** (1) If $D_x f(x;\bar{t})$ or $[D_x f(x;\bar{t})]^T$ is an invertible $B_0(B)-$matrix for all $x \in X$, then $D_x f(x;\bar{t})$ is a $P_0(P)-$matrix (with $\det D_x f(x;\bar{t}) > 0$ in the case of a $P_0$-matrix) by Lemma 3. Thus, $f : X \times T \to \mathbb{R}^n$ is globally univalent by Theorems 4 and 4w in Gale and Nikaido (1965).

(2) The set $\mathcal{E}$ is a finite set since $\mathcal{E}$ is compact (it is a closed subset of the compact space $M$) and discrete ($f$ is one-to-one in a neighborhood of each $\bar{x} \in \mathcal{E}$ by the IFT). Then the zeros of $M$ are isolated, so the Poincaré-Hopf Theorem implies that the index sum is equal to the Euler characteristic of the unit disk, which is $+1$. This establishes existence. Since $\det(D_x f(\bar{x};\bar{t})) > 0$ at each $\bar{x} \in \mathcal{E}$, it follows that the index of $f$ at each $\bar{x} \in \mathcal{E}$ is $+1$, and this establishes uniqueness. ■

**Remark 1** *The assumption that $f$ "points outward" on the boundary of $M$ means that for any $x$ on the boundary of $M$, there exists a sequence $\varepsilon_i \downarrow 0$ such that $x + \varepsilon_i f(x;\bar{t}) \notin M$ for $i = 1, 2, 3....$*

**Remark 2** *The second part of Theorem 3 relies on topological properties, so it applies as long as $X$ is open and $M$ is replaced everywhere with a set $M' \subset X$ which is diffeomorphic to the closed unit disk.*

### 3.1 The Role of Heterogeneity

This section illustrates various aspects of the comparative statics result. A heuristic discussion gives intuition as to how ill-behaved comparative statics may arise in the presence of heterogeneous interaction terms; an analytic example illustrates how the mean positive dominance condition captures the trade-off between heterogeneity and magnitude while also demonstrating the role played by the rows and columns of the inverse of the Jacobian; and a formal result makes precise how the interaction terms can be larger in magnitude when there is less heterogeneity while still retaining the mean positive dominance property.

To begin heuristically, consider a three equation system, $f = (f_1, f_2, f_3)$. The normalized Jacobian,

$$\Lambda D_x f(x;t) = \begin{bmatrix} 1 & f_{12}/f_{11} & f_{13}/f_{11} \\ f_{21}/f_{22} & 1 & f_{23}/f_{22} \\ f_{31}/f_{33} & f_{32}/f_{33} & 1 \end{bmatrix},$$

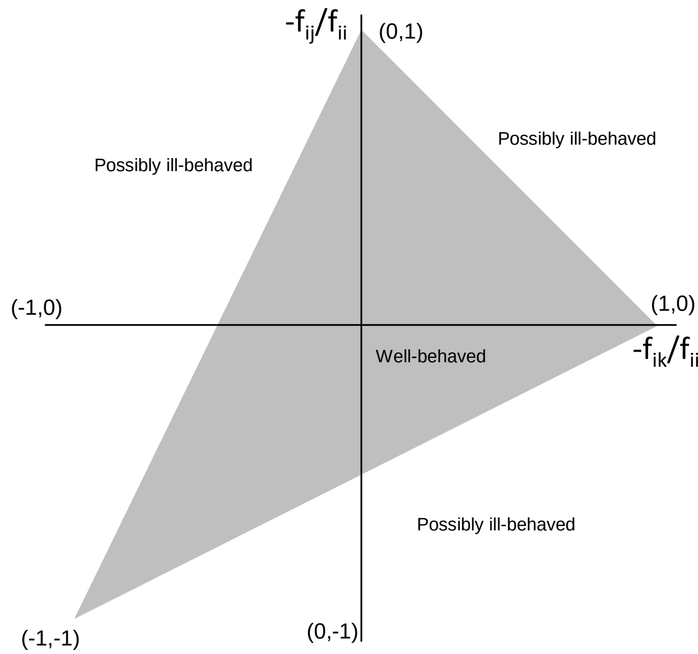


Figure 1: The mean positive dominance condition for row $i$ of $\Lambda Df_x(x;t)$ when $n=3$.

is a $B_0-$matrix if, for each row $i=1,2,3$,

$$1+f_{ij}/f_{ii}+f_{ik}/f_{ii} \geq \max\{0, 3f_{ij}/f_{ii}, 3f_{ik}/f_{ii}\} \text{ for } i \neq j \neq k.$$

These conditions are illustrated for a single row $i$ by the triangle in Figure 1 with vertices at $(0,1)$, $(1,0)$, and $(-1,-1)$. If there is at least one row where the interaction terms are outside of this set then the normalized Jacobian is not a $B-$matrix, and hence comparative statics may be ill-behaved.

Recalling the discussion in the introduction, ill-behaved comparative statics seem plausible when the interaction effect is large or if there is heterogeneity in the interaction terms. The intuition behind the size of negative interaction effects is obvious, so let me focus on the role of heterogeneity.

To use a congestion goods example, consider a recreational activity like downhill skiing or surfing. In each setting, the location (a ski resort or wave) is fixed in the short run. Assume the marginal utility of the activity for any skier or surfer is decreasing with congestion.[14] Additional participants means one is more likely

[14] We can allow for positive externalities from companions as long as some strangers generate

to be in a collision, to have more difficulty skiing a clean run or catching a good wave, and the interval between runs or rides is longer because of congestion at the lift or line-up. However, skilled participants may generate smaller externalities since these participants are more knowledgeable of etiquette and less likely to interfere with one's enjoyment of the activity. If willingness to pay for the activity is positively related with skill, then skilled participants may be willing to pay more to be among a greater quantity of skilled participants as opposed to a lesser quantity of unskilled participants.[15] Thus, a ski resort may be able to charge a higher price and attract more skiers if it can select for more highly skilled skiers. This may explain why resorts with more difficult trails can seem higher priced and more crowded than equally sized, nearby resorts with easier trails.[16]

The following analytic example illustrates the role of heterogeneity, the role of inverse row and column sums, and the role of mean positive dominance in partial effects for individual level comparative statics.

**Example 1** Consider a market for a social good with three consumers. As in equation (4) the system of demand functions is

$$f(x;p) \equiv x - g(x;p) = 0.$$

Suppose that at equilibrium the Jacobian, which is identical to the normalized Jacobian in this case, is given by

$$D_x f(\bar{x};\bar{p}) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

with $a < 0$, $b > 0$, and $c > 0$. Consumer $1's$ demand is not influenced by others' consumption. Consumer $2's$ demand is increasing in consumer $1's$ consumption[17] but

negative externalities. This case would be situated in the upper left or lower right quadrants of Figure 1. The pure congestion case is represented in the lower left quadrant.

[15] In this example, system $f$ is a demand system and the interaction terms represent an individual's marginal demand response to a one unit increase in other's consumption of the good.

[16] This example does not apply well to all congestion situations. Probably the most reasonable assumption for traffic congestion is anonymous effects since in the vast majority of cases each additional vehicle creates same negative externality. In this case the demand curve is downward sloping. That being said, it may be possible for toll operators to select for better drivers, and consequently face a less elastic demand curve, by selling passes only to those who have good driving records.

[17] Although it may seem counterintuitive, it follows from equation (3) that interaction terms have

independent of consumer $3's$ consumption, and consumer $3's$ demand is decreasing in both consumer 1 and consumer $2's$ consumption.

Assume that the partial price effect for each consumer $i$ is nonpositive, $\frac{\partial g_i}{\partial p} \leq 0$ for $i = 1, 2, 3$. Under what conditions is market demand downward sloping? What about individual demand?

For the reader's convenience, note that

$$Dx(\bar{p}) = [D_x f(\bar{x}; \bar{p})]^{-1} (-D_p f(\bar{x}; \bar{p})) = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac - b & -c & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial p} \\ \frac{\partial g_2}{\partial p} \\ \frac{\partial g_3}{\partial p} \end{bmatrix}.$$

The **slope of market demand** is

$$\frac{d\bar{\Sigma}}{dp} = \sum \frac{dx_i}{dp} = \frac{\partial g_1}{\partial p}(1 - a - b + ac) + \frac{\partial g_2}{\partial p}(1 - c) + \frac{\partial g_3}{\partial p}, \tag{9}$$

where the coefficient on the $jth$ element of $-D_p f(\bar{x}; \bar{p})$ is the $jth$ column sum of $[D_x f(\bar{x}; \bar{p})]^{-1}$.

Focus on the first term of expression (9). If consumer 1 decreases consumption by one unit, consumer 2 decreases consumption by $-a$ units while consumer 3 increases consumption by $b$ units. Moreover, consumer 2's decrease in consumption causes consumer 3 to decrease consumption by an additional $-ac$ units. Thus, $\frac{\partial g_1}{\partial p}(1 - a - b + ac)$ represents the contribution of consumer 1's partial effect to the slope of market demand after other consumers fully respond to his change in consumption. The intuition behind the other terms of (9) is analogous.

The slope of market demand clearly depends on the size of the interaction effects of consumers 1 and 2 on consumer 3, but it also depends on their variation. To see this, suppose $a = -0.5$ and $b = c = 0.8$. In this case $D_x f(\bar{x}; \bar{p})$ is a $B-$matrix and its first inverse column sum is 0.3. But if $b$ and $c$ change to 1.6 and 0, respectively, then $D_x f(\bar{x}; \bar{p})$ is no longer a $B-$matrix[18] and its first inverse column sum is -0.1, even though the cumulative interaction effect $b + c = 1.6$ remains constant.

Note that since the diagonal dominance condition depends only on the absolute row sums, it is unable to distinguish between these two cases. Clearly, the Jaco-

a positive influence on the total effect if $-f_{ij}/f_{ii} \geq 0$ and a negative influence if $-f_{ij}/f_{ii} \leq 0$.

[18] In both cases, the row sum of the third row is 2.6. But the largest positive off-diagonal term in the first case is 0.8 while in the second it is 1.6. Since $n = 3$, we have $2.6 > 3 \times 0.8$ but $2.6 < 3 \times 1.6$.

bian is not diagonally dominant in either case. Moreover, the demand system is not monotone, so one is unable to apply the results from the MCS literature, either.

Turning to the **slope of individual demand**, note that

$$\frac{dg_3}{dp} = \frac{\partial g_1}{\partial p}(ac - b) + \frac{\partial g_2}{\partial p}(-c) + \frac{\partial g_3}{\partial p}. \tag{10}$$

When the price increases, the partial effect on consumer 1's consumption affects consumer 3's consumption directly $(-b)$ and indirectly through its effect on consumer 2's consumption $(ac)$. Similarly, the change in consumer 2's consumption affects consumer 3's consumption directly $(-c)$. In principle, there could also be an indirect effect through consumer 1, but since $f_{12} = 0$, consumer 1's demand is unresponsive to changes in consumer 2's consumption. Finally, a change in consumer 3's consumption could indirectly affect his own consumption through its effect on others, but this requires feedback effects and the parametric assumptions rule this out.

As with the slope of market demand, limiting the size and heterogeneity of interaction effects is important to ensure that individual demand is downward sloping. However, equation (10) indicates that we also need a restriction on the size and heterogeneity of other consumers' partial responses relative to consumer 3's partial response. This explains why Theorem 1 imposes the mean positive dominance condition on the vector of partial effects.

Finally, observe the slope of market demand depends on how a change in own consumption *affects* others' consumption, but the slope of individual demand depends on how own consumption is *affected by* changes in other's consumption. The off-diagonal row entries of the normalized Jacobian capture the former force, while the off-diagonal column entries capture the latter. This explains why the aggregate comparative statics results in part 1 of Theorem 1 places a mean positive dominance restriction on the rows of the matrix while the individual level results in part 2 places the same restriction on its columns. ■

Turning to formal results, the mean positive dominance condition is satisfied if the "spread" between the largest and smallest interaction terms of a row is bounded. Moreover, this bound must be made smaller as the interaction terms become larger.

To see this, define for each row $i = 1, ..., n$,

$$\begin{aligned} f_i^+ &= \max\{0, -f_{ij}/f_{ii} \mid j \neq i\} \text{ and} \\ f_i^- &= \min\{0, -f_{ij}/f_{ii} \mid j \neq i\} \end{aligned}$$

as the largest positive and smallest negative interaction term, respectively. The *multiplicity of $f_i^-$ given $i$*, denoted by $r_i$, is the number of interaction terms in the *ith* row equal to $f_i^-$, where we define $r_i = 0$ if there are no strictly negative interaction terms. Then we have the following result.

**Proposition 1** *Let $n \geq 2$. Suppose there are $0 \leq k_i \leq n-1$ strictly positive interaction terms in row $i$. Then the normalized Jacobian $\Lambda D_x f(x;t)$ is a $B_0(B)-$matrix if, for all $i = 1, ..., n$,*

$$f_i^- \geq (>) - \frac{1}{n - r_i} + \frac{k_i}{n - r_i} f_i^+. \tag{11}$$

**Proof.** For a given $i$, let $\alpha_i \in \{1, ..., n\} \backslash i$ be the set of indices $j$ (excluding $i$) such that $-\frac{f_{ii}}{f_{ij}} > 0$ is a strictly positive interaction term, and let $\alpha_i^c \in \{1, ..., n\} \backslash i$ be its complement. Then $1 + \sum_{j \in \alpha_i \cup \alpha_i^c} \frac{f_{ij}}{f_{ii}} = 1 - \sum_{j \in \alpha_i} \left(-\frac{f_{ij}}{f_{ii}}\right) - \sum_{j \in \alpha_i^c} \left(-\frac{f_{ij}}{f_{ii}}\right) \geq 1 - k_i f_i^+ - r_i f_i^-$. Thus, the (strict) row mean positive dominance condition is satisfied when, for all $i = 1, ..., n$,

$$1 - k_i f_i^+ - r_i f_i^- \geq (>) - n f_i^-.$$

Rearrange this inequality to get (11). ■

Inequality (11) illustrates the trade-off between heterogeneity and magnitude in several ways. To begin, assume there no strictly positive interaction terms ($k_i = 0$, $f_i^+ = 0$) and interaction is anonymous in row $i$ in the sense that $-\frac{f_{ij}}{f_{ii}} = -\frac{f_{ik}}{f_{ii}}$ for all $j \neq k \neq i$. Then $r_i = n - 1$ and the right hand side of inequality (11) is $-1$. If one increases heterogeneity by allowing for nonanonymous interaction terms, then this lower bound increases to $-\frac{1}{n - r_i}$, which is decreasing in $r_i$. If one increases heterogeneity yet again by allowing for some strictly positive interaction terms ($k_i > 0$), then the lower bound increases again given $r_i$. Finally note that if all interactions are nonnegative ($k_i = n-1$, $r_i = 0$, $f_i^- = 0$), then inequality (11) reads $f_i^+ < \frac{1}{n-1}$. These observations are in accord with the $n = 3$ case case presented in Figure 1.

Proposition 1 also makes it easy to see that the results in this paper generalize Dixit (1986), since in that paper the off-diagonals of the normalized Jacobian are assumed identical by rows (see equation 39-ii in Dixit). In this context, diagonal dominance requires the absolute value of each off-diagonal term to be less than $\frac{1}{n-1}$. Inequality (11) delivers precisely the same bound for positive interaction effects and the much weaker bound of $-1$ for negative interaction effects. Thus, part 1 of Theorem 1 provides the same results in Dixit[19] under weaker conditions.[20] The reason we are able to generalize Dixit is that, in contrast to the $B-$matrix property, diagonal dominance fails to take advantage of the limit on heterogeneity imposed by assuming interaction effects are identical by rows.

# 4 Applications

In this section I apply the results of the paper to optimization problems, differentiable games, and the smooth competitive exchange model. These applications illustrate that it is simple to check whether the Jacobian is a $B-$matrix, and that requiring the Jacobian to be a $B-$matrix results in economically meaningful restrictions. The approach also simplifies the analysis in some cases and provides new insights in others.

## 4.1 Optimization Problems

Consider the parameterized problem

$$\text{Maximize } F(x;t) \text{ subject to } x \in X, \tag{12}$$

where $F : X \times T \to \mathbb{R}$ is twice continuously differentiable, $X \subset \mathbb{R}^n$, and $T \subset \mathbb{R}$.

To apply the results of this paper, let $f(x;t) \equiv \nabla F(x;t)$ be the gradient of $F$ and observe that $D_x f(x;t)$ is the Hessian of $F$. The following observations and definitions are then natural.

**Lemma 5** *Fix $t \in T$. The $C^2$ function $F : \mathbb{R}^n \to \mathbb{R}$ is (strictly) convex at $x$ if $D_x f(x;t)$ is a $B_0(B)-$matrix. The $C^2$ function $F : \mathbb{R}^n \to \mathbb{R}$ is (strictly) concave at $x$ if $-D_x f(x;t)$ is a $B_0(B)-$matrix.*

[19] See the discussion after equation (41-ii) in Dixit (1986).

[20] In an effort to place meaningful restrictions on comparative statics, Dixit (1986) constrains his analysis to equilibria which are stable under tâtonnement dynamics. See the application to aggregative games below for more on the relationship between comparative statics and stability in this environment.

**Proof.** This follows directly from the facts that a symmetric $B-$matrix is positive definite, and that a symmetric $B_0-$matrix is positive semidefinite (Lemma 3). ■

**Definition 1** The $C^2$ function $F : \mathbb{R}^n \to \mathbb{R}$ is *(strictly) $B-$convex* if $D_x f(x;t)$ is a $B_0(B)-$matrix. The $C^2$ function $F : \mathbb{R}^n \to \mathbb{R}$ is *(strictly) $B-$concave* if $-D_x f(x;t)$ is a $B_0(B)-$matrix.

Recall that $F(x;t)$ has *increasing differences in* $(x,t)$ iff $\frac{\partial^2 F}{\partial x_i \partial t} \geq 0$ for all $i = 1, ..., n$, or, equivalently, iff $D_t f(x;t) \geq 0$. I can now state the main result for unconstrained optimization problems.

**Proposition 2** *Let $F : X \times T \to \mathbb{R}$ be $C^2$, where $T \subset \mathbb{R}$ and $X \subset \mathbb{R}^n$ are open sets. Given $\bar{t}$, let $\bar{x} = x(\bar{t}) = \arg\max_{x \in X} F(x;\bar{t})$ be interior and unique. Suppose that $F(\bar{x};\bar{t})$ is $B-$concave and* $\det D_x f(\bar{x};\bar{t}) \neq 0$.

1. *The equilibrium aggregate $H(\bar{\Sigma})$ is nondecreasing in $t$ whenever $F$ has increasing differences in $(\bar{x},\bar{t})$.*
2. *$\bar{x}_i$ is nondecreasing in $t$ whenever $D_t f(\bar{x};\bar{t})$ is mean positive dominant in element $i$. If $F$ is strictly $B-$concave at $(\bar{x};\bar{t})$ and $D_t f(\bar{x};\bar{t})$ is strictly mean positive dominant in element $i$, then $\bar{x}_i$ is strictly increasing in $t$.*

**Proof.** A necessary condition for a local maximum is $f(\bar{x};\bar{t}) = 0$. Since $F$ is $B-$concave, $-D_x f(x;t)$ is a $B-$matrix. By symmetry of the Hessian, $[-D_x f(x;t)]^T$ is a $B-$matrix, too. The result follows from Theorem 1 applied to equation (2). ■

Proposition 2 facilitates comparison with the classic results in the MCS literature. It is well-known from this literature that the highest and lowest element from the set of solutions $\arg\max_{x \in X} F(x;t)$ is nondecreasing in $t$ if and only if $F$ is supermodular[21] in $x$ and has increasing differences in $(x,t)$. Proposition 2 instead addresses the comparative statics of the aggregate and individual components of the solution vector when $F$ is $B-$concave, but may not be supermodular in $x$ and may not necessarily have increasing differences in $(x,t)$. In other words, Proposition 2 applies in non-monotone environments.

[21] Recall that $F(x;t)$ is *supermodular in* $x$ iff $f_{ij} \geq 0$ for all $j \neq i$ and all $i = 1, ..., n$, or, equivalently, iff the off diagonals of the Hessian $D_x f(x;t)$ are nonnegative.

### 4.1.1 Platform Monopoly

The results in Proposition 2 can be used to generalize the comparative statics results in Rochet and Tirole's (2003) seminal study of the two-sided platform monopoly to allow for multiple sides and more general volume functions. A platform firm facilitates interactions between third parties like buyers and sellers. The credit card and game console industries are examples.

For a multi-sided monopoly, the volume of transactions is given by $Q(p)$, where $p = (p_1, ..., p_n)$ is the vector of prices that the platform firm charges side $i$ per transaction. The monopolist's optimization problem is

$$\max_p \pi(p; c, t) = \left( \sum_i^n p_i - c \right) Q(p; t),$$

where $c$ is the cost per transaction to the firm and $t$ is a volume shift parameter.[22]

In the Rochet and Tirole ($RT$) two-sided version of this problem the sides are buyers ($B$) and sellers ($S$). The volume is given by

$$Q(p_B, p_S; t) = D^B(p_B; t) D^S(p_S; t) \tag{13}$$

where the quasi-demand functions $D^i(p_i; t) = \Pr(b_i \geq p_i; t)$ for $i = B, S$ are the proportion of buyers (respectively, sellers) for whom the gross surplus $b_i$ is higher than the price $p_i$.

$RT$ analyze the comparative statics of marquee and captive buyers. *Marquee buyers* are represented by a small uniform shift in sellers' surpluses so that

$$D^S(p_S; t) = D^S(p_S - t).$$

*Captive buyers* have perfectly inelastic demand for the platform, perhaps because they are tied by long-term contracts, so the buyers' quasi-demand function is

$$D^B(p_B; t) = D^B(p_B) + t.$$

$RT$ show that if the quasi-demand functions are log concave, the seller price $p_S$

[22] This version of the profit function can also accommodate membership fees; see Rochet and Tirole (2006).

increases when there are marquee buyers while the buyer price $p_B$ increases when there are some captive buyers. It also follows from their analysis that the total price total price per transaction collected by the monopolist, $p_B + p_S$, increases with both captive and marquee buyers.

These findings are limited by the fact that the buyer and seller quasi-demand functions are multiplicatively separable and independent of the other's price. The next result demonstrates that Proposition 2 can be used to generalize the $RT$ log concavity condition to a multi-sided model with a general volume function. I assume $\frac{\partial^2 \pi}{\partial p_i \partial p_j} \leq 0$ for some $j \neq i$ for two reasons: (1) if $\frac{\partial^2 \pi}{\partial p_i \partial p_j} \geq 0$ for all $j \neq i$ then the profit function is supermodular and the results from the MCS literature apply, and (2) $\frac{\partial^2 \pi}{\partial p_i \partial p_j} \leq 0$ is probably the empirically relevant case.

**Proposition 3** *Given $\bar{c}$ and $\bar{t}$, let $\bar{p} = \arg\max_{p \geq 0} \pi(p; \bar{c}, \bar{t})$ be unique and strictly positive. At the maximizer $\bar{p}$, assume, for all $i = 1, ..., n$, that $\frac{\partial Q}{\partial p_i} < 0$, $\frac{\partial^2 \pi}{\partial p_i \partial p_j} \leq 0$ for some $j \neq i$, and $\frac{1}{n}\left(\frac{\partial^2 Q}{\partial p_i^2} + \sum_{k \neq i} \frac{\partial^2 Q}{\partial p_k \partial p_i}\right) \leq \frac{\partial^2 Q}{\partial p_j \partial p_i}$ for all $j \neq i$.*

1. *Then the total price per transaction, $\sum p_i$, increases with $t$ whenever $\pi(\bar{p}; \bar{c}, \bar{t})$ has increasing differences in $(p; t)$.*

2. *If, in addition, $\frac{\partial^2 \pi}{\partial p_i \partial t} > 0$ and $\frac{\partial^2 \pi}{\partial p_j \partial t} = 0$ for all $j \neq i$, then we may also conclude that $p_i$ increases with $t$. More generally, $p_i$ increases with $t$ whenever $D_t \nabla \pi(\bar{p}; \bar{c}, \bar{t})$ is mean positive dominant in element $i$.*

The key condition in Proposition 3 is that, for each side $i$, $\frac{1}{n}\left(\frac{\partial^2 Q}{\partial p_i^2} + \sum_{k \neq i} \frac{\partial^2 Q}{\partial p_k \partial p_i}\right) \leq \frac{\partial^2 Q}{\partial p_j \partial p_i}$ for $j \neq i$. If we think of $\frac{\partial Q}{\partial p_j}$ as the side-$j$ marginal volume, this condition means that the average effect on side-$j$ marginal volume of an increase in the price for side-$i$ is less than its effect on any particular side $k$ different from $i$. Roughly, changing the price on one side of the market does not have too dramatic or heterogeneous of an effect on the pricing incentives that the monopoly faces in other sides of the market.

It turns out that the $RT$ log concavity conditions are a special case of the conditions provided in Proposition 3.

**Corollary 1** *Suppose $n = 2$ and the volume function is the $RT$ volume function given by (13). Evaluating the expressions below at the maximizer $\bar{p}$ we have*

1. $\frac{\partial^2 Q}{\partial p_i^2} \leq \frac{\partial^2 Q}{\partial p_i \partial p_j}$ *for $i, j \in \{S, B\}$ $i \neq j$ if and only if $D^B(p_B; t)$ and $D^S(p_S; t)$ are log concave,*

2. $\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} \leq 0$ *for* $i, j \in \{S, B\}$ $i \neq j$, *and*

3. *with captive buyers we have* $\frac{\partial^2 \pi}{\partial p_B \partial t} > 0$ *and* $\frac{\partial^2 \pi}{\partial p_B \partial t} = 0$; *with marque buyers,* $\frac{\partial^2 \pi}{\partial p_S \partial t} \geq 0$ *and* $\frac{\partial^2 \pi}{\partial p_B \partial t} = 0$ *if* $D^S(p_S; t)$ *is log concave.*

#### 4.1.2 The Consumer Problem

Theorem 1 can also be applied to the comparative statics of constrained optimization problems like the classic utility maximization problem. Assuming Walras' Law, then given an $n-$vector of prices $p$, income $m > 0$, and $C^2$ utility $u : \mathbb{R}^n_+ \to \mathbb{R}$, the consumer solves

$$\max_x u(x) \text{ s.t. } \sum_i p_i x_i = m. \tag{14}$$

Our interest is in the conditions on utility under which all goods are normal. This question can be addressed by applying the tools in this paper to the familiar system of first order conditions:

$$\begin{aligned} \frac{u_i(\bar{x})}{p_i} &= \lambda \text{ for } i = 1, ..., n, \text{ and} \\ \sum_i p_i \bar{x}_i &= m. \end{aligned}$$

Totally differentiating the first $n$ first order conditions with respect to $m$, denoting the matrix $[U(x)/p] = (u_{ij}/p_i)$ and the column vectors $dx = (\frac{dx_1}{dm}, ..., \frac{dx_n}{dm})$, $d\lambda = \left(\frac{d\lambda}{dm}, ..., \frac{d\lambda}{dm}\right)$, we have

$$[U(\bar{x})/p]dx = d\lambda. \tag{15}$$

Given $d\lambda$, this system has a unique solution whenever the Hessian of $u$ at $\bar{x}$, denoted $U(\bar{x}) = (u_{ij}(\bar{x}))$, is nonsingular. If, in addition, $p^T U^{-1} p \leq 0$ for all $p$, it follows that

$d\lambda < 0$ for all $p$.[23] So to apply the results of the paper, rewrite (15) as

$$[-U(\bar{x})/p]dx = -d\lambda$$

Since the vector $-d\lambda$ is positive and uniform it follows from Theorem 1 that all goods are normal if $[-U(\bar{x})/p]^T$ is mean positive dominant. This requires, for all goods $i = 1, ..., n$,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{u_{ki}(\bar{x})}{p_k} \leq \min\left\{0, \frac{u_{ki}(\bar{x})}{p_k}|k \neq i\right\}. \tag{16}$$

The term $\frac{u_{ji}(\bar{x})}{p_j}$ is the change in the marginal utility per dollar spent on good $j$ when the consumption of good $i$ increases by one. Thus, the interpretation of condition (16) is that the average effect on the marginal utility per dollar spent on each good of an increase in consumption of good $i$ is nonpositive and no greater than the effect on the marginal utility per dollar spent on any particular good $j$ different from $i$.

A different interpretation is available which involves only the utility function. Dropping the dependence on $\bar{x}$ and making use of the fact that $\frac{u_i}{u_j} = \frac{p_i}{p_j}$ at a maximizer, rewrite these conditions as, for $i = 1, ...n$,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{u_{ki}(\bar{x})}{u_k} \leq \min\left\{0, \frac{u_{ki}}{u_k}|k \neq i\right\}. \tag{17}$$

The term $\frac{u_{ji}}{u_j}$ is the percentage change in the marginal utility of good $j$ caused by a unit increase in the consumption of good $i$. Then condition (17) states that for a unit increase in good $i$, the average percentage change in the marginal utility across all goods should not be positive and should be no greater than the percentage change in

[23] From the first order conditions and the IFT we have

$$\begin{bmatrix} U & -p \\ -p^T & 0 \end{bmatrix}\begin{bmatrix} dx \\ \frac{d\lambda}{dm} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Denoting $\mathcal{H} = \begin{bmatrix} U & -p \\ -p^T & 0 \end{bmatrix}$, applying Cramer's rule, and using the Laplacian expansion to evaluate the numerator, we have $\frac{d\lambda}{dm} = -\frac{\det U}{\det \mathcal{H}}$. Since $U$ is nonsingular it follows that $\frac{d\lambda}{dm} \neq 0$. Using the Schur complement method to calculate the determinant, and noting that the Schur complement of $\mathcal{H}$ with respect to $U$ is $\mathcal{H}/U = -p^TU^{-1}p$, we have $\det \mathcal{H} = \det U \det \mathcal{H}/U = -p^TU^{-1}p \det U$. Hence, $\frac{d\lambda}{dm} = (p^TU^{-1}p)^{-1}$.

the marginal utility of any particular good different from $i$. Note that this interpretation is invariant under linear transformations of utility. The following proposition offers a formal statement of these results.

**Proposition 4** *Consider the utility maximization problem (14) with prices $p >> 0$ and income $m > 0$. Assume Walras' Law is satisfied, there is a unique (global) maximizer $\bar{x} >> 0$, $u$ is $C^2$, and $U(\bar{x})$ is nonsingular. If condition (17) holds at $\bar{x}$ for $i = 1, ..., n$, then all good are normal, that is, $\frac{d\bar{x}_i}{dm} \geq 0$ for $i = 1, ..., n$.*

**Proof.** Under condition (17) and the nonsingularity of $U(\bar{x})$, Theorem 1 implies $dx \gtreqless 0$ as $d\lambda \lesseqgtr 0$. But $dx < 0$ contradicts Walras' Law, so it follows that $d\lambda \leq 0$. ■

The conditions of Proposition (4) imply that $p^T U^{-1} p \leq 0$ at $\bar{x}$ since $\frac{d\lambda}{dm} = (p^T U^{-1} p)^{-1}$ (see footnote 23). Note that this inequality is also satisfied when $u$ is concave.[24]

Chipman (1977) and Quah (2007) both assume that utility satisfies some version of concavity and that all goods are complements to ensure that all goods are normal. When utility is $C^2$, complementarity means $u_{ij} \geq 0$ for all $i \neq j$. This assumption implies that the incentive to consume a good increases with the consumption of every other good.

Concavity in these paper plays an important role because of the constraint. To see this, imagine there are three goods with strong complementarities between goods 2 and 3 but none between good 1 and the other two. When income increases, the "partial" incentive to consume *all* goods may increase (in the sense that $u_{ii} < 0$ for all $i$), but it may be optimal for the consumer to reduce consumption of good 1 to take advantage of the strong complementarities between goods 2 and 3. Concavity places restrictions on the cross partials which rule out this scenario. Note that this intuition suggests that concavity is not necessary in the $n = 2$ case with complements, a fact which was proven in Amir (2005).

In contrast to the complementarity assumption in Chipman (1977) and Quah (2007), condition (17) allows for some goods (possibly all) to be substitutes, or $u_{ij} < 0$ for some $i \neq j$. In this paper and theirs, when a constraint is present it seems that

[24] Recall that concavity requires $U$ to be negative semidefinite for all $x$, or $z^T U z \leq 0$ for all $x, z \in \mathbb{R}^n$. Thus, at values of $x$ where $U$ is invertible, all the eignevalues of $U$ are strictly negative. The eigenvalues of $U^{-1}$ are the inverses of the eigenvalues of $U$, so $U^{-1}$ is negative definite at values of $x$ (including $\bar{x}$) where $U$ is invertible. This implies $p^T U^{-1} p < 0$.

the important provision driving normality is that the cross effects on marginal utility are limited. To wit, note that both the conditions provided here and in Chipman (1977) and Quah (2007) are satisfied if preferences are strongly separable so that they may be represented by an additively separable utility function, in which case $u_{ij} = 0$ for all $i \neq j$.

## 4.2 Games With Differentiable Payoffs

Consider a parameterized non-cooperative game $\Gamma_t = ((\pi_i, X_i)_{i \in \mathcal{N}}, t)$ with a finite set of players $\mathcal{N} = \{1, ..., n\}$, one-dimensional convex strategy sets $X_i \subset \mathbb{R}$, and $C^2$ strictly quasiconcave payoff functions $\pi_i : X \times T \to \mathbb{R}$. An equilibrium is defined in the usual way: $\bar{x}(t) = (\bar{x}_1(t), ..., \bar{x}_n(t))$ is a Nash equilibrium if, for each player $i$, $\bar{x}_i(t) \in \arg\max_{x_i \in X} \pi_i(x_i, \bar{x}_{-i}, t)$.

Given the strategy profile of the other players, a necessary condition for $\bar{x}_i$ to be a maximizer is

$$\frac{\partial \pi_i(\bar{x})}{\partial x_i} \leq 0 \text{ with equality if } \bar{x}_i \in int(X_i).$$

And for $\bar{x}_i \in int(X_i)$, the second order necessary condition is $\frac{\partial^2 \pi_i(\bar{x})}{\partial x_i^2} < 0$ since strict quasiconcavity implies $\bar{x}_i$ is unique.

The subsequent analysis can accommodate equilibria where $\bar{x}_i \in bdy(X_i)$, but for the sake of brevity concentrate on interior equilibria. Define the function $F(x;t) \equiv \pi(x;t) = (\pi_1(x;t), ..., \pi_n(x;t))$. Then the gradient of $F$ is $f(x,t) = \nabla \pi(x;t)$,[25] and at any interior equilibrium $\bar{x}$ we have $f(\bar{x};\bar{t}) = 0$. The Jacobian of $F$ is then the matrix of second derivatives of the profit functions for each player:

$$D_x f(x;t) = \begin{bmatrix} \frac{\partial^2 \pi_1}{\partial x_1^2} & \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 \pi_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi_2}{\partial x_2^2} & & \frac{\partial^2 \pi_2}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 \pi_n}{\partial x_n \partial x_1} & \frac{\partial^2 \pi_n}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 \pi_n}{\partial x_n^2} \end{bmatrix},$$

Note that $D_x f(x;t)$ is not mean positive dominant since it has a negative diagonal.

[25] Rosen (1965) calls this the pseudogradient of the game.

However, $-D_x f(x;t)$ is mean positive dominant if and only if

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\partial^2\pi_i}{\partial x_i \partial x_j} \leq \min\left\{0, \frac{\partial^2\pi_i}{\partial x_i \partial x_j} \mid j \neq i\right\} \text{ for all } i. \tag{18}$$

The interpretation is that the average effect on player $i's$ marginal payoff of an equal increase in each player's strategy is nonpositive and less than the effect of an increase in any single player $j's$ strategy on player $i's$ $(j \neq i)$ marginal payoff.

Alternatively, $[-D_x f(x;t)]^T$ is mean positive dominant if and only if

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2\pi_i}{\partial x_i \partial x_j} \leq \min\left\{0, \frac{\partial^2\pi_i}{\partial x_i \partial x_j} \mid j \neq i\right\} \text{ for all } j. \tag{19}$$

Here the interpretation is that the average effect on players' marginal payoff of an increase in player $j's$ strategy is nonpositive and less than its effect on the marginal payoff of any particular player $i's$ $(i \neq -j)$ marginal payoff. Note that $D_x f(x;t)$ is not mean positive dominant since it has a strictly negative diagonal. Then the following results are implied by Theorems 1-3 without additional proof.

**Proposition 5** *Consider the game $\Gamma_t$ and assume that an interior equilibrium $\bar{x}$ exists given $\bar{t}$.*

1. *If (18) holds at equilibrium, then the equilibrium aggregate, $H\left(\sum \bar{x}_i\right)$, is increasing in $t$ whenever payoffs have increasing differences in $(x,t)$ : $\frac{\partial^2\pi_i}{\partial x_i \partial t} \geq 0$ for all $i$. Moreover, player $i's$ strategy $\bar{x}_i$ increases with $t$ whenever the parameter increases player $i's$ marginal payoff but has no direct effect on other players: $\frac{\partial^2\pi_i}{\partial x_i \partial t} > 0$ and $\frac{\partial^2\pi_j}{\partial x_j \partial t} = 0$ for $j \neq i$.*

2. *Suppose (19) holds at equilibrium. Then player $i$'s strategy, $\bar{x}_i$, is increasing in $t$ whenever the effect of an increase in $t$ on marginal payoffs is mean positive dominant for player $i$, that is, whenever $D_t \nabla\pi(\bar{x};\bar{t})$ is mean positive dominant in element $i$.*

3. *Suppose that either (a) (18) holds and $D_x f(\bar{x};\bar{t})$ is symmetric at equilibrium $\bar{x}$, or (b) (18) or (19) holds at equilbrium and there is a constant speed of adjustment. Then equilibrium is locally asymptotically stable under tâtonnement dynamics.*

*4. Suppose (18) or (19) holds (with strict inequality) for all $x \in X$ and $X$ is an open (closed) rectangle. Then there is at most one equilibrium.*

Proposition (5) can be used to generalize the Cournot oligopoly comparative statics results in Dixit (1986) and Corchón (1994). In the standard model, firm $i$ chooses output $x_i \geq 0$ to maximize profit $\pi(x_i, x_{-i}; t) = x_i P(X; t) - c_i(x_i; t)$, where $X = \sum x_i$. Both papers identify conditions under which an idiosyncratic positive shock to firm $i$ (e.g., a decrease in firm $i's$ marginal cost) raises firm $i's$ output and industry output. Evaluated at equilibrium, the union of these conditions is that, for all $i$,

$$\begin{array}{ll} \frac{dP}{dx_i} - \frac{d^2c_i}{dx_i^2} \leq -n\left(\frac{dP}{dx_i} + \bar{x}_i\frac{d^2P}{dx_i^2}\right) & \text{if } \frac{dP}{dx_i} + \bar{x}_i\frac{d^2P}{dx_i^2} \geq 0 \text{ and} \\ \frac{dP}{dx_i} - \frac{d^2c_i}{dx_i^2} \leq 0 & \text{if } \frac{dP}{dx_i} + \bar{x}_i\frac{d^2P}{dx_i^2} < 0. \end{array}$$

The first of these conditions comes from Dixit and the second from Corchón. Observing that $\frac{\partial^2\pi_i}{\partial x_i \partial x_j} = \frac{dP}{dx_i} + x_i\frac{d^2P}{dx_i}$, these conditions apply when a firm's rivals are strategic complements and, respectively, strategic substitutes. Since $P(x) - \frac{dc_i}{dx_i}$ is the markup over marginal cost, these conditions place restrictions on the impact of an increase in own output on this markup. It is a straightforward exercise to check that these are exactly the inequalities implied by condition (18), and hence the conclusion of Part 1 of Proposition (5) applies.[26]

The main contribution of Proposition (5) to the literature on Cournot oligopoly is that it generalizes the Dixit-Corchón conditions to the differentiated products case where firms choose $x_i \geq 0$ to maximize $\pi(x_i, x_{-i}) = x_i P_i(x_1, ..., x_n; t) - c_i(x_1, ..., x_n; t)$. Notice that each firm may have a different inverse demand function and firms costs can depend on others' output. This specification is quite general and could serve as a model of monopolistic competition, research and development, strategic relationships between upstream and downstream firms, among others.

[26] It is also worth pointing out that these conditions are precisely the conditions which one obtains when taking the union of the stability conditions identified by Hahn (1962) and Seade (1980). In addition, under the condition that $\frac{dP}{dx_i} - \frac{d^2c_i}{dx_i^2} < 0$, Kolstad and Mathieson (1987) show that uniqueness obtains if and only if

$$\sum_{i=1}^{n} \frac{\frac{dP}{dx_i} + x_i\frac{d^2P}{dx_i}}{\frac{dP}{dx_i} - \frac{d^2c_i}{dx_i^2}} > -1.$$

This inequality is satisfied when condition (18) is met.

Recently, Acemoglu and Jensen (2013) exploit the aggregative structure of Cournot oligopoly to weaken the Dixit-Corchón conditions for well-behaved comparative statics in the standard Cournot model. Their results have a limited application in the differentiated products case, however, since the approach requires the payoffs and marginal payoffs of each firm to depend on the output of other firms in the same way through the same aggregator function.

## 4.3 General Equilibrium

Following the textbook treatment (Chapter 17 of Mas-Colell, Whinston, and Green, 1995), consider a competitive exchange economy with $L+1$ goods and $N$ consumers. Let $p = (p_1, ..., p_{L+1}) \in \mathbb{R}_+^{L+1}$ be a vector of prices, let $\omega_k = (\omega_{k1}, ..., \omega_{k(L+1)}) \in \mathbb{R}_+^{L+1}$ be consumer $k's$ endowment vector, and let $y_k : \mathbb{R}_+^{L+1} \times \mathbb{R}_+^{L+1} \to \mathbb{R}_+^{L+1}$ be consumer $k's$ Walrasian demand function and $y_{k\ell} : \mathbb{R}_+^{L+1} \times \mathbb{R}_+^{L+1} \to \mathbb{R}_+$ consumer $k's$ demand for good $\ell$. Then consumer $k's$ excess demand for good $\ell$ is $y_{k\ell}(p, p \cdot \omega_k) - \omega_{k\ell}$, and the aggregate excess demand for good $\ell$ is $f_\ell(p; \omega) = \sum_{k=1}^{N} (y_{k\ell}(p, p \cdot \omega_k) - \omega_{k\ell})$. Given endowments $\omega = (\omega_1, ..., \omega_N)$, aggregate excess demand is $f(p; \omega) = (f_1(p; \omega), ..., f_{L+1}(p; \omega))$. Normalizing the price of good $L+1$ to one, we can represent the exchange economy with the aggregate excess demand equations for the first $L$ goods:

$$\hat{f}(p; \omega) = (\hat{f}_1(p; \omega), ..., \hat{f}_L(p; \omega)).$$

Given endowments $\bar{\omega}$, an equilibrium price vector $p(\bar{\omega})$ is defined as the points where $\hat{f}$ vanishes, $\hat{f}(p(\bar{\omega}); \bar{\omega}) = 0$.

If the excess demand functions satisfy the gross substitutes property,[27] it is well-known that the equilibrium is unique, stable under tâtonnement dynamics, and has nice comparative statics properties. On the last point, one objective of comparative statics analysis in general equilibrium theory is to find conditions under which the relative price of good $\ell$ decreases when the endowment of the same good increases for the $kth$ consumer. The answer to this question is complicated by the fact that a change in the endowment of one good directly impacts the excess demands for all other goods through wealth effects. But the desired relationship holds when excess demand satisfies the gross substitutes property and consumer $k$'s excess demand for

[27] Recall that the gross substitutes property requires $\frac{\partial f_\ell}{\partial p_j} \geq 0$ for all $j = 1, ..., L+1$, $j \neq \ell$, and $\ell = 1, ..., L+1$. In other words, the matrix $D_p f(p; \omega)$ has nonnegative off-diagonal elements.

all goods is normal.

Mean positive dominance in partial price effects provides an alternative to the gross substitutes property and allows for complements and inferior goods. Note that compared to the notation in the rest of the paper, the endogenous vector is now $p$ instead of $x$, and the parameter is now $\omega_{k\ell}$ instead of $t$. We are thus interested in signing elements of the vector

$$Dp(\omega) = -\left[D_p\hat{f}(p;\omega)\right]^{-1} D_\omega\hat{f}(p;\omega).$$

Then we have the following result.

**Proposition 6** *Assume* $\hat{f}_{\ell\ell}(\bar{p};\bar{\omega}) < 0$ *for all* $\ell = 1, ..., L+1$.

1. *If* $-D_p\hat{f}(\bar{p};\bar{\omega})$ *is an invertible* $B_0-$*matrix, and all non-numeraire goods are normal (inferior) for consumer* $k$, *then the average price level is nondecreasing (nonincreasing) in consumer* $k'$*s endowment of the numeraire:* $\frac{d}{d\omega_{k(L+1)}}\left(\frac{1}{L}\sum_{\ell=1}^{L} p_\ell(\omega)\right) \geq (\leq) 0$.

2. *If* $[-D_p\hat{f}(\bar{p};\bar{\omega})]^T$ *is an invertible* $B_0-$*matrix and* $-D_{\omega_{k\ell}}\hat{f}(\bar{p};\bar{\omega})$ *is mean positive dominant in element* $\ell$ *then the relative price of good* $\ell$ *is nonincreasing in consumer* $k'$*s endowment of good* $\ell$ : $\frac{d\bar{p}_\ell}{d\omega_{k\ell}} \leq 0$.

3. *Suppose that the excess demand function for each good is positive as its price goes to zero. If, for each equilibrium* $\bar{p}$, $-D_p\hat{f}(\bar{p};\bar{\omega})$ *or* $[-D_p\hat{f}(\bar{p};\bar{\omega})]^T$ *is a* $B-$*matrix, then there is exactly one equilibrium.*

4. *If* $[-D_p\hat{f}(\bar{p};\bar{\omega})]^T$ *and* $-D_p\hat{f}(\bar{p};\bar{\omega})$ *are* $B-$*matrices, then equilibrium is locally asymptotically stable under tâtonnement dynamics with a constant speed of adjustment:* $c_i = c$ *for all* $i$.

**Proof.** (1) Let $m_k = p \cdot \omega_k$ be consumer $k's$ wealth and note that when the endowment of the numeraire increases for the $kth$ consumer, the typical element of the vector $D_\omega\hat{f}(\bar{p};\bar{\omega}) = D_{\omega_{k(L+1)}}\hat{f}(\bar{p};\bar{\omega})$ is $\frac{\partial y_k(\bar{p};\bar{m}_k)}{\partial m_k}p_{L+1} = \frac{\partial y_k(\bar{p};\bar{m}_k)}{\partial m_k}$, which is nonnegative iff $\frac{\partial y_k(\bar{p};\bar{m}_k)}{\partial m_k} \geq 0$. The result follows from Theorem 1.

(2) Define the change of variable $t = -\omega_{\ell k}$. If $[-D_p\hat{f}(\bar{p};\bar{\omega})]^T$ is a $B-$matrix and $D_t\hat{f}(\bar{p};\bar{\omega}) = -D_{\omega_{k\ell}}\hat{f}(\bar{p};\bar{\omega})$ is mean positive dominant in element $\ell$, then $\frac{dp_\ell}{dt} \geq 0$ by Theorem 1.

(3) Without loss of generality we may normalize prices to be confined to the price simplex as in Dierker (1972). The result then follows from Varian (1975) since the determinant of a $B-$matrix is positive. Varian's proof is a version of part 2 of Theorem 3, and the desirability assumption ensures that $-f$ points outward on the boundary of the price simplex.

(4) This is a direct application of Theorem 2. ■

To interpret Proposition 6, note that when consumer $k's$ demand for non-numeraire goods is normal, the partial effect of an increase in consumer $k's$ endowment of the numeraire causes his excess demand for the numeraire to fall and the excess demand for all other goods to increase. Consequently, the partial effect lowers the nominal price of the numeraire and raises the nominal price of all other goods. On the other hand, if the demand for all non-numeraire goods is inferior, by Walras' Law the demand for the numeraire must increase by at least as much as the endowment increases. In this case, the partial effect of the endowment shock lowers the price of the other goods and increases the price of the numeraire. Part 1 of Proposition 6 asserts that after all the general equilibrium effects are accounted for, the $B-$matrix property imposed on $-D_p\hat{f}(\bar{p};\bar{\omega})$ ensures that the partial effects are maintained on average. Of course, any good can be chosen as the numeraire, so this result hold quite broadly.

The condition that $-D_p\hat{f}(\bar{p};\bar{\omega})$ is a $B_0-$matrix makes exact the idea that the demand for a good must be most responsive to changes in its own price compared to changes in the price of other goods. Precisely, mean positive dominance requires that, for all $\ell = 1, ..., L$,

$$\hat{f}_{\ell\ell} \leq \min\left\{0, L\hat{f}_{\ell j} | j \neq \ell\right\} - \sum_{j\neq\ell, j=1}^{L} \hat{f}_{\ell j}, \tag{20}$$

In words, condition (20) says that the own partial price effect, $\hat{f}_{\ell\ell} < 0$, must be larger in magnitude (by the margin $\min\left\{0, L\hat{f}_{\ell j} | j \neq \ell\right\}$) than the cumulative partial effect of a change in the price of each of the other goods on the demand for good $\ell$, $\sum_{j\neq\ell, j=1}^{L} \hat{f}_{\ell j}$.

Part 2 of Proposition 6 requires careful interpretation. If the endowment of good $\ell$ increases for consumer $k$, choose some other good $i \neq \ell$ as the numeraire. The assumption that $-D_{\omega_{k\ell}}\hat{f}(\bar{p};\bar{\omega})$ is mean positive dominant in element $\ell$ reads[28]

$$1-\frac{\partial y_{k\ell}(\bar{p};\bar{m}_k)}{\partial m_k}p_\ell \geq -L\phi+\sum_{j\neq\ell,j=1}^{L}\frac{\partial y_{kj}(\bar{p};\bar{m}_k)}{\partial m_k}p_\ell, \tag{21}$$

where $\phi\equiv\min\left\{0,\frac{\partial y_k(\bar{p};\bar{m}_k)}{\partial m_k}p_\ell|j\neq\ell\right\}$. The terms $\frac{\partial y_{kj}(\bar{p};\bar{m}_k)}{\partial m_k}p_\ell$ represent the increase in consumption of good $j$ due to the fact that wealth increases by $p_\ell$ when the endowment of good $\ell$ increases by one. Thus, the left hand side of inequality (**??**) is the increase in the excess supply of good $\ell$ when prices remain constant. The sum $\sum_{j\neq\ell,j=1}^{L}\frac{\partial y_{kj}(\bar{p};\bar{m})}{\partial m}p_\ell$ is the cumulative change in the excess demand for goods $j\neq\ell$. Thus, inequality (**??**) requires that the increase in the excess supply of good $\ell$ is larger (by the margin $-L\phi$) than the cumulative change in the excess demand for other goods, holding prices constant. This implies that the price of good $\ell$ decreases "significantly" compared to the change the prices of other non-numeraire goods.

Then the assumption that $[-D_p\hat{f}(\bar{p};\bar{\omega})]^T$ is a $B_0-$matrix ensures that the price of good $\ell$ decreases relative to the numeraire after all the general equilibrium effects are realized. This condition is analogous to requiring that $-D_p\hat{f}(\bar{p};\bar{\omega})$ is a $B_0-$matrix, except that it applies to the columns instead of the rows. This makes precise the idea that if the price of a good $\ell$ changes, the strongest partial demand response must be from the demand for good $\ell$. Note that part 2 of Proposition 6 implies only that that price of good $\ell$ decreases relative to the numeraire; the stronger conclusion that the price of good $\ell$ decreases relative to *all* other goods would require that the conditions of part 2 apply independently of which good $j\neq\ell$ is chosen as the numeraire.

Unlike results which rely on the gross substitutes property, conditions (20) and (**??**) allow for some inferior goods and some complements. On the other hand, if the gross substitutes property is satisfied then of course condition (20) is more restrictive. The $B-$matrix property also delivers a strong uniqueness result, but the stability conclusion is weaker than that obtained via the gross substitutes property.

A few additional comparative statics results exist for general equilibrium. Nachbar (2002) identifies a way to normalize prices such that comparative statics are well-behaved for a given profile of endowment changes and the associated consumption changes. Proposition 6 is a more traditional comparative statics result in that,

[28] $m_k = p\cdot\omega_k$ is consumer $k's$ wealth.

given a price normalization, it finds observable conditions on the excess demand functions under which an exogenous shock causes the endogenous variables to move in the desired direction. Malinvaud (1972, end of Chapter 5.2) provides the desired comparative statics result in a distribution economy under the assumption that the law of demand holds. In contrast to an exchange economy in which income depends on endowments, tough, a distribution economy takes income as fixed.

A different strand of the general equilibrium comparative statics literature focuses on the refutability of general equilibrium theory under the rationality assumption. Roughly speaking, the Sonnenschien-Mantel-Debreu theorem states that there exists an economy that can rationalize any observed set of prices, and hence general equilibrium theory can not be refuted by data on prices alone. Brown and Matzkin (1996) show, however, that if endowments and prices are observable, then the general equilibrium model is refutable. Brown and Shannon (2000) demonstrate that, in a smooth setting, given finite data on prices, aggregate endowments, and incomes, these data can be rationalized in the general equilibrium model if and only if they can be rationalized in a general equilibrium model in which equilibrium is locally unique, locally stable, and that comparative statics are locally well-behaved. One interpretation of this result is that the smooth general equilibrium model is refutable only under stronger assumptions. Proposition 6 provides such conditions.

# 5 Conclusion

This paper makes three key contributions. First, it illuminates the role played by the heterogeneity of interaction terms in determining the behavior of an economic model, specifically as it pertains to comparative statics, stability and uniqueness. Many existing comparative statics in the literature are made possible through assumptions which limit heterogeneity. However, these assumptions are often considered primitives in the models so it is not clear how heterogeneity affects the results. The mean positive dominance condition captures a trade-off between heterogeneity and magnitude.

Second, this paper introduces the class of $B-$ and $B_0-$matrices to economics. It is simple to check whether the Jacobian belongs to this class, and imposing this structure on an economic model generates economically meaningful restrictions. Also, this class of matrices possesses many attractive technical properties.

Third, Theorems 1 and 2 provide a novel statement, albeit a weak version, of

Samuelson's (1947) correspondence principle for multidimensional models that are not necessarily monotone. While Samuelson focused on the duality between comparative statics and stability, Theorems 1-3 suggest the presence of a triality which includes uniqueness.

On a final note, understanding why unexpected outcomes occur is a particularly important charge for our field. A focus on the conditions which generate well-behaved models provides insight on this question as it identifies necessary conditions for unexpected outcomes. For example, this paper shows that if there are at least some negative interaction effects, and interaction effects are sufficiently large or heterogeneous, then even at stable equilibria comparative statics may be ill-behaved.

# 6 Appendix

PROOF OF LEMMA 1

(1) Since $y = A^{-1}b$ we have for each $i$, $y_i = \sum_{j=1}^{n} b_j \delta_{ij}$. It follows that

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \sum_{j=1}^{n} b_j \delta_{ij} = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \delta_{ij} \right) b_j \geq 0,$$

where the last inequality is valid for all $b \geq 0$ iff $A \in \mathcal{NICS}$.

(2) This follows from the fact that $y_i = \sum_{j=1}^{n} b_j \delta_{ij} = b_i \delta_{ii}$ when $b$ is positive only in element $i$.

(3) In this case, $b_i = \bar{b} \geq 0$ for all $i$, so $y_i = \sum_{j=1}^{n} b_j \delta_{ij} = \bar{b} \left( \sum_{j=1}^{n} \delta_{ij} \right) \geq 0$, where the last inequality is valid for all $\bar{b} > 0$ iff $A \in \mathcal{NIRS}$. ■

PROOF OF LEMMA 3

(1) This follows from the facts that a $B-$matrix has a strictly positive determinant, the determinant is a continuous function of the matrix entries, and weak inequalities are preserved in the limit.

(2) The proof of this claim is a slight modification of the proof of Proposition 2.5 in Peña (2001). Consider the set of natural numbers $M = \{1, ..., n\}$ ; let $\alpha$ be a subset of $M$ with $k$ elements, and let $\alpha'$ be the complement of $\alpha$ in $M$, $\alpha' = M \backslash \alpha$. The elements of $\alpha$ and $\alpha'$ are understood to be arranged in increasing order. Denote by $A[\alpha]$ any principal submatrix of $A$ with elements $(a_{ij})$ with $i, j \in \alpha$.

First we establish that $A[\alpha]$ has nonnegative row sums. To this end, note that $a_{ii} \geq \sum_{h \in H} |a_{ih}|$ , where $H = \{h | 1 \leq h \leq n \text{ and } a_{ih} < 0\}$ . This follows from the fact that rearranging (7) implies $a_{ii} - a_i^+ \geq \sum_{j \neq i} (a_i^+ - a_{ij})$ and observing that $a_{ii} \geq a_{ii} - a_i^+$, $a_i^+ - a_{ij} \geq 0$ for all $i \neq j$, and $a_i^+ - a_{ij} \geq |a_{ij}|$ if $a_{ij} < 0$.

To show that $ka_{ij} \leq \sum_{s \in \alpha} a_{is}$ for all $i \neq j \in \alpha$, assume instead that for some $i \neq j \in \alpha$, $ka_{ij} > \sum_{s \in \alpha} a_{is}$ to derive a contradiction. Since $ka_i^+ \geq ka_{ij}$ for all $i$, it follows that for some $i$,

$$na_i^+ \geq ka_i^+ + \sum_{r \in \alpha'} a_{ir} > \sum_{s \in \alpha} a_{is} + \sum_{r \in \alpha'} a_{ir} = \sum_{p=1}^{n} a_{ip},$$

which contradicts the assumption that $A$ is a $B_0-$matrix.

(3)-(4) These are straightforward corollaries of (2).

(5) From equation (7) it follows that if $a_i^+ = 0$, then $a_{ii} \geq 0$. If $a_i^+ > 0$, then since $\sum_{j=1}^n a_{ij} \leq a_{ii} + (n-1)a_i^+$, equation (7) implies $a_{ii} + (n-1)a_i^+ \geq na_i^+$, or $a_{ii} \geq a_i^+$.

(6) $A+A^T$ is a symmetric $B(B_0)-$matrix by Lemma 2, so it is positive (semi)definite from Peña (2001) and part (4). $A$ is positive (semi)definite if and only if $A + A^T$ is positive (semi)definite, and similarly for $A^T$.

(7) Again, $A$ is positive (semi)definite if and only if $A+A^T$ is positive (semi)definite, and similarly for $A^T$.

(8) This follows from Proposition 2.3 and Theorem 4.3 in Peña (2001). In an effort to keep this paper reasonably self-contained, I will state the key parts of these results here. Let $A = (a_{ik})_{1\leq i,k\leq n}$ be a real matrix. Define, for each $i = 1, ..., n$,

$$r_i^+ = \max\{0, a_{ij}|j \neq i\} \text{ and } c_i^+ = \max\{0, a_{ij}|i \neq j\}.$$

Then define, for each $i = 1, ..., n$,

$$\theta_i = \min\left\{a_{ii} - r_i^+ - \sum_{k\neq i}\left(r_i^+ - a_{ik}\right), a_{ii} - c_i^+ - \sum_{k\neq i}\left(c_i^+ - a_{ki}\right)\right\}.$$

Letting $\lambda$ be an eigenvalue of $A$, the first part of Peña's (2001) Theorem 4.3 implies

$$\text{Re}(\lambda) \geq \min\{\theta_1, ..., \theta_n\}.$$

Proposition 2.3 in Peña (2001) states that the real matrix $A$ is a $B-$matrix if and only if for all $i = 1, ..., n$,

$$a_{ii} - r_i^+ - \sum_{k\neq i}\left(r_i^+ - a_{ik}\right) > 0.$$

It follows that $A^T$ is a $B-$matrix if and only if for all $i = 1, ..., n$,

$$a_{ii} - c_i^+ - \sum_{k\neq i}\left(c_i^+ - a_{ki}\right) > 0.$$

Consequently, $\min\{\theta_1, ..., \theta_n\} > 0$ if $A$ and $A^T$ are $B-$matrices. ■

## Proof of Proposition 3

The necessary first order conditions for a maximum are

$$\frac{\partial \pi}{\partial p_i} = \left(\sum p - c\right) \frac{\partial Q}{\partial p_i} + Q = 0 \text{ for all } i.$$

To apply Proposition 2, we need to show that the Hessian of $-\pi$ is mean positive dominant. The first order condtions imply that $\frac{\partial Q}{\partial p} \equiv \frac{\partial Q}{\partial p_i}$ for all $i$ at a maximum, so the terms along the main diagonal of the Hessian of $-\pi$ are $-\frac{\partial^2 \pi}{\partial p_i^2} = -\left(\sum p - c\right) \frac{\partial^2 Q}{\partial p_i^2} - 2\frac{\partial Q}{\partial p}$. The off diagonal terms are $-\frac{\partial^2 \pi}{\partial p_i \partial p_j} = -\left(\sum p - c\right) \frac{\partial^2 Q}{\partial p_i \partial p_j} - 2\frac{\partial Q}{\partial p}$. The remainder of the proof exploits the fact that $\frac{\partial^2 \pi}{\partial p_i \partial p_j} = \frac{\partial^2 \pi}{\partial p_j \partial p_i}$.

Since $-\frac{\partial^2 \pi}{\partial p_i \partial p_j} \leq 0$ for some $j \neq i$, .the Hessian (of $-\pi$) is mean positive dominant if, for $i = 1, ..., n$,

$$\left(\sum p - c\right) \left( \frac{\partial^2 Q}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 Q}{\partial p_j \partial p_i} \right) + 2n \frac{\partial Q}{\partial p} \leq n \left(\sum p - c\right) \frac{\partial^2 Q}{\partial p_j \partial p_i} + n2 \frac{\partial Q}{\partial p} \quad \forall j \neq i,$$

which simplifies to $\frac{\partial^2 Q}{\partial p_i^2} + \sum_{j \neq i} \frac{\partial^2 Q}{\partial p_j \partial p_i} \leq n \max \left\{ \frac{\partial^2 Q}{\partial p_j \partial p_i} | j \neq i \right\}_i^+$, as desired. We have now shown that the profit function is $B-$concave under the conditions in Proposition 3, so Proposition 2 applies. ■

## Proof of Corollary 1

(1) Note that for $i, j = \{S, B\}$ and $j \neq i$,

$$\frac{\partial^2 Q}{\partial p_i^2} = \frac{\partial^2 D^i}{\partial p_i^2} D^j \quad \text{and} \quad \frac{\partial^2 Q}{\partial p_i \partial p_j} = \frac{\partial D^i}{\partial p_i} \frac{\partial D^j}{\partial p_j}.$$

The first order conditions are $\left(\sum p - c\right) \frac{\partial Q}{\partial p_i} + Q = 0$ for all $i$ which implies $D^i D^j = -\left(\sum p - c\right) \frac{\partial D^i}{\partial p_i} D^j$ for $i, j \in \{B, S\}$ and $i \neq j$. Thus, $\frac{D^S}{\frac{\partial D^S}{\partial p_S}} = \frac{D^B}{\frac{\partial D^B}{\partial p_B}}$ in equilibrium. It follows that $\frac{\partial^2 Q}{\partial p_i^2} \leq \frac{\partial^2 Q}{\partial p^i \partial p^j}$ for $i \neq j \in \{B, S\}$ if and only if, for $i \in \{B, S\}$,

$$\frac{\partial^2 D^i}{\partial p_i^2} D^j \leq \frac{\partial D^i}{\partial p_i} \frac{\partial D^j}{\partial p_j}, \quad \text{or} \quad \frac{\partial^2 D^i}{\partial p_i^2} D^i \leq \left( \frac{\partial D^i}{\partial p_i} \right)^2.$$

This last condition is precisely log-concavity of the quasi-demand functions.

(2) $\frac{\partial\pi}{\partial p_B \partial p_S} = \frac{\partial D^B}{\partial p_B} D^S + D^B \frac{\partial D^S}{\partial p_S} + (\sum p - c) \frac{\partial D^B}{\partial p_B} \frac{\partial D^S}{\partial p_S} = D^B \frac{\partial D^S}{\partial p_S} \leq 0$, where the second inequality follows from the first order conditions.

(3) With captive buyers, we have $\frac{\partial^2\pi}{\partial p_B \partial t} = D^S > 0$ and $\frac{\partial^2\pi}{\partial p_S \partial t} = (\sum p - c) \frac{\partial D^S}{\partial p_s} + D^S = 0$ by the first order condition, so we conclude that $p^B + p^S$ and $p^B$ increase with more captive buyers.

With marquee buyers, $\frac{\partial^2\pi}{\partial p_B \partial t} = -\frac{\partial D^S(p-t)}{\partial p_S} \left( D^B(p_B) + (\sum p - c) \frac{\partial D^B}{\partial p_B} \right) = 0$ using the first order conditions. Once again using the first order conditions and the assumption that $D^S$ is log concave, we have

$$
\begin{aligned}
\frac{\partial^2\pi}{\partial p_S \partial t} &= -D^B(p_B) \left( \frac{\partial D^S(p-t)}{\partial p_S} + \left( \sum_i^n p_i - c \right) \frac{\partial^2 D^S(p-t)}{\partial p_S^2} \right) \\
&= -D^B(p_B) \left( \frac{\partial D^S(p-t)}{\partial p_S} - \frac{D^S}{\frac{\partial D^S}{\partial p_S}} \frac{\partial^2 D^S(p-t)}{\partial p_S^2} \right) \\
&\geq 0.
\end{aligned}
$$

Thus, $p^B + p^S$ and $p^S$ increase with the addition of marquee buyers.[29]

[29] $RT$ also show that the seller price decreases with more captive buyers while the buyer price decreases with the prevalence of marquee buyers. In the two-sided case we can generate these results via Cramer's Rule under the additional condition that $\frac{\partial^2\pi}{\partial p_j \partial t} = 0$.

# References


[1] Acemoglu, Daron and Martin Kaae Jensen (2013) Aggregate Comparative Statics. Games and Economic Behavior, 27-49.

[2] Amir, Rabah (2005) Supermodularity and Complementarity in Economics: An Elementary Survey. Southern Economic Journal, 71(3), 636-660.

[3] Amir, Rabah and Val E. Lambson (2000) On the Effects of Entry in Cournot Markets. Review of Economic Studies, 67: 235-254.

[4] Brown, Donald J. and Rosa L. Matzkin (1996) Testable Restrictions on the Equilibrium Manifold. Econometrica, 64(6): 1249-1262.

[5] Brown, Donald J. and Chris Shannon (2000) Uniqueness, Stability, and Comparative Statics in Rationalizable Walrasian Markets. Econometrica, 68(6): 1529-1539.

[6] Carnicer, J. M., T. N. T. Goodman, and J. M. Peña (1999) Linear Conditions for Positive Determinants. Linear Algebra and Its Applications 292 (1-3): 39-59.

[7] Chipman, John (1977) An Empirical Implication of Auspitz-Lieben-Edgeworth-Pareto Complementarity. Journal of Economic Theory 14: 228-231.

[8] Dierker, Egbert (1972) Two Remarks on the Number of Equilibria of an Economy. Econometrica, 40(5): 951-953.

[9] Dixit, Avinash (1986) Comparative Statics for Oligopoly. International Economic Review, 27(1): 107-122.

[10] Gale, David and Hukukane Nikaido (1965) The Jacobian Matrix and Global Univalence of Mappings. Mathematische Annalen 159(2): 81-93.

[11] Jinji, Naoto (2014) Comparative Statics of Oligopoly: A Generalized Result. Economics Letters 124: 79-82.

[12] Kehoe, Timothy, J. (1980) An Index Theorem for General Equilibrium Models with Production, Econometrica. 48: 1211-1232.

[13] Kolstad, Charles D. and Lars Mathiesen (1987) Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium, The Review of Economic Studies. 54(4): 681-690.

[14] Malinvaud, E. *Lectures on Microeconomic Theory.* London: North-Holland.

[15] Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995) *Microeconomic Theory* Oxford University Press, Oxford.

[16] Milgrom, Paul and John Roberts (1990) The Economics of Modern Manufacturing: Technology, Strategy, and Organization. The American Economic Review, 80(3): 511-528.

[17] Milgrom, Paul and Chris Shannon (1994) Monotone Comparative Statics, Econometrica 62(1): 157-180.

[18] Monaco, Andrew J. and Tarun Sabarwal (forthcoming) Games with Strategic Complements and Substitutes. Economic Theory.

[19] Monderer, Dov and Lloyd S. Shapley (1996) Potential Games. Games and Economic Behavior, 14: 124-143.

[20] Nachbar, John H. (2002) General Equilibrium Comparative Statics, Econometrica 70(5): 2065-2074.

[21] Peña, J. M. (2001) A class of $P-$Matrices with Applications to the Localization of the Eigenvalues of a Real Matrix. SIAM Journal on Matrix Analysis and Applications 22(3): 1027-1037.

[22] Pollak, Robert A. (1976) Interdependent Preferences, American Economic Review, 66(3): 309-320.

[23] Quah, John K.-H. (2007) The Comparative Statics of Constrained Optimization Problems. Econometrica 75(2): 401-431.

[24] Rosen, J. B. (1965) Existence and Uniqueness of Equilibrium Points for Concave N-Person Games, Econometrica, 33(3): 520-534.

[25] Roy, Sunanda and Tarun Sabarwal (2008) On the (Non-)Lattice Structure of the Equilibrium Set in Games with Strategic Substitutes, Economic Theory, 37: 161-169.

[26] Roy, Sunanda and Tarun Sabarwal (2010) Monotone Comparative Statics for Games with Strategic Substitutes. Journal of Mathematical Economics 46: 793–806.

[27] Samuelson, Paul (1947) *Foundation of Economic Analysis.* Oxford University Press, London.

[28] Simon, Carl P. (1989) Some Fine-Tuning for Dominant Diagonal Matrices, Economics Letters, 30: 217-221.

[29] Topkis, Donald M. (1998) *Supermodularity and Complementarity.* Princeton University Press.

[30] Varian, Hal R. (1975) A Third Remark on the Number of Equilibria of an Economy, Econometrica, 43(5/6): 985-986.

[31] Villas-Boas (1997) Comparative Statics of Fixed Points, Journal of Economic Theory, 73: 183-198.