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Comparative Statics, Stability, and Uniqueness

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Abstract

Consider an economic model whose equilibrium can be represented as the fixed point of a system of differentiable equations. Using the theory of $B-$matrices, I show that comparative statics are well-behaved if the interactions between the equations are not too large, and the negative interactions are not too varied. When there are only positive interactions, for example when strategic complements prevail in a strategic setting, I prove a version of Samuleson's (1947) Correspondence Principle in that equilibrium is nondecreasing for any positive parameter shock if and only if equilibrium is exponentially stable under discrete time best reply dynamics. If there are only negative interactions, like when strategic substitutes prevail in a game theoretic context, I use the theory of inverse $M-$matrices to significantly relax Dixit's (1986) conditions under which comparative statics are well-behaved. For every comparative statics result I show that if the conditions apply globally then equilibrium is unique. Applications are provided to differentiated products Cournot oligopoly, market demand with interdependent preferences, and games on fixed networks.

Keywords: Correspondence Principle, Interdependent Preferences, Oligopoly, Networks, M-matrices, inverse M-matrices, B-matrices, $P-$matrices, global univalence

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1 Introduction

In many economic models, the equilibrium values of endogenous variables can be characterized as the solution to a system of equations. Of paramount interest is the equilibrium effect on the endogenous variables of a change in an exogenous parameter. Using recent results from linear algebra and the familiar Implicit Function Theorem approach, I provide novel results and insights on this canonical problem.

While the results apply to any system of equations, the intuition can be best understood in a game theoretic context where the system is interpreted as the set of best response functions, one for each player. The total equilibrium effect of a positive shock on a player's action can be decomposed into a partial effect and an interactions effect. The partial effect is the increase in the player's action holding constant all other player's actions. The interactions effect is the difference between the total effect and partial effect. If there are negative interaction effects, meaning that there is a player whose best response is decreasing in the action of some other player, then the private effect and interactions effect may have opposite signs. Consequently, the total effect and private effect may have opposite signs as well.

Comparative statics are "well-behaved," meaning that the total effect and private effect have the same sign, when interaction effects are moderate and the negative interaction effects are not too varied. Specifically, if the column (row) means of the Jacobian are positive and larger than each of its off-diagonal elements, I show that its inverse has nonnegative row (column) sums. This is sufficient for well-behaved comparative statics. The result builds off of Carnicer, Goodman and Peña (1999) who show that these matrices, termed $B-$matrices in Peña (2001), have a strictly positive determinant. The class of $B-$matrices is distinct from diagonally dominant matrices and appears to be new to the economics literature.

To put this result in context, Dixit (1986) studies comparative statics in a homogeneous products Cournot oligopoly with strategic substitutes. Demand depends only on the sum of output, so interaction is anonymous. Dixit shows that comparative statics are well-behaved under a diagonal dominance condition, which with anonymous interaction requires each of the off-diagonal terms in the (normalized) Jacobian to be less than $\frac{1}{n-1}$. My result demonstrates that, for a wider class of parameter shocks, well-behaved comparative statics arise under the same condition if either interaction is anonymous *or* there are strategic substitutes.

When strategic substitutes *and* anonymous interaction exist, I exploit the theory of inverse $M-$matrices to show that Dixit's condition can be relaxed to $\frac{1}{\sqrt{n-2}}$. I also show that diagonal dominance is sufficient for well-behaved comparative statics with (non-anonymous) strategic substitutes.

When strategic complements prevail, so that all interaction effects are positive, ill-behaved comparative statics are possible with large interaction effects. However, in accord with Samuelson's (1947) Correspondence Principle, equilibrium is nondecreasing for any positive shock if and only if equilibrium is exponentially stable. This is an Implicit Function Theorem-based version of Echenique's (2002) lattice-based result in which he shows that if the equilibrium is not monotone increasing then equilibrium is unstable. I also show that equilibrium is unique if the spectral radius of the normalized Jacobian is everywhere less than one.

More generally, for each result, if the conditions on the (normalized) Jacobian which guarantee well-behaved comparative statics apply globally then equilibrium is unique, if it exists. All but one of the comparative statics results involve conditions under which the Jacobian is a $B-$matrix, an $M-$matrix, or an inverse $M-$matrix in equilibrium. Since each of these classes is a type of $P-$matrix, Gale and Nikaido's (1965) global univalence result gives uniqueness if the Jacobian is everywhere a $B-$matrix, an $M-$matrix, or an inverse $M-$matrix. The remaining comparative statics result (Theorem 5) requires a norm of the Jacobian to be less than one at equilibrium. If the norm is everywhere less than one, uniqueness follows from Lemma 3 in Christensen and Jung (2010).

A unifying principal which emerges from the analysis is that heterogeneity matters. Well-behaved comparative statics arise at stable equilibria when interaction effects are not too varied. But if there are at least some negative interaction effects, and interaction effects are sufficiently heterogeneous, then even at stable equilibria comparative statics may be ill-behaved. I provide plausible examples of this possibility in the context of a differentiated products Cournot oligopoly (Example 3) and market demand with interdependent preferences (Section 7).

In fact, the application in Section 7 helps unify the analysis in Leibenstein (1950), Rohlfs (1974) and Becker (1991) within a simple two-person model of market demand with interdependent preferences. In another application of the results, I characterize comparative statics in the network game with linear best replies studied in Bramoullé, Kranton, and D'Amours (2014) in terms of a player's degree.

The analysis in this paper contains and extends the results in Dixit (1986). Corchón (1994) extends Dixit by considering aggregative games where payoffs depend on other's actions only through the sum. Acemoglu and Jensen (2013) consider a more general class of aggregative games, but one restriction which remains is that the interaction effects of other players on own action must have the same sign. When applied to their respective settings, my results are complementary to theirs. Like this paper, Jinji (2014) considers a general interaction environment. Within the context of oligopoly, Jinji derives conditions under which the equilibrium effect of a unit change in another player's *action* has the same sign as the private effect. In addition to diagonal dominance, Jinji imposes a condition on certain minors of the Jacobian which is difficult to interpret.

In contrast, the linear conditions on the Jacobian in this paper are simple to check and interpret, and they convey key intuition about the problem. If the system characterizing equilibrium is derived from payoff maximization, then with some caveats like interiority, the conditions on the Jacobian translate directly into conditions on the second order derivatives of the objective function. Finally this paper additionally contributes to the literature by linking comparative statics and uniqueness.

A recent strand of the comparative statics literature focuses on environments in which the Implicit Function Theorem cannot be applied. Early contributions to this literature include Topkis (1998), Vives (1990), and Milgrom and Roberts (1990) who use lattice-based techniques to study monotonicity of equilibrium in games with strategic complements. However, these results apply to only extremal equilibria and the techniques do not extend to other types of strategic interaction. A few results But see Roy and Sabarwal (2010) and Acemoglu and Jensen (2013) for some results in games with strategic substitutes which do not require smoothness. Monaco and Sabarwal (2015) also provide some results which require continuity but not differentiability.

One final point of distinction from the existing literature is that assumptions in this paper are made directly on the equations of the equilibrium system. This makes the results widely applicable. For example, they apply to reduced form macroeconomic models as well as strategic environments with expected payoff maximizers.

In the next section I present the model. The main comparative statics result is in Section 3. I characterize stability in Section 4. In Section 5 I study the relationship between comparative statics and stability, especially in the cases of strategic complements and strategic substitutes. The linear case is considered in Section 6, and Section 7 is devoted to an extended application to demand with interdependent preferences. Uniqueness results appear throughout. Section 8 concludes.

2 The Environment

Consider a system of n equations in n unknowns:

$$\begin{aligned} f^1(x_1, x_2, ..., x_n; t) &= 0 \\ f^2(x_1, x_2, ..., x_n; t) &= 0 \\ \vdots \quad & \quad \vdots \\ f^n(x_1, x_2, ..., x_n; t) &= 0 \end{aligned} \tag{1}$$

where $x_i \in X_i \subseteq \mathbb{R}$ for all i are the endogenous variables and $t \in T \subseteq \mathbb{R}^s$ is a vector of exogenous parameters. For the sake of clarity and interpretation, the main body of the paper assumes $s = 1$, but the results generalize to s finite.

An equilibrium is a vector $x^* = (x_1^*, x_2^*, ..., x_n^*)$ that satisfies system (1). Since our interest is in comparative statics, and existence theorems abound for such a system, assume directly that an equilibrium exists. In addition assume

1. $f = (f^1, f^2, ..., f^n)$ is continuously differentiable at x^*, and
2. $\det(\tilde{A}) \neq 0$ at x^*, where $\tilde{A}$ is the Jacobian of f.
3. $\frac{\partial f^i}{\partial x_i} \neq 0$ for all i at x^*.

The first two assumptions ensure that I can apply the Implicit Function Theorem. The last assumption allows for a convenient normalization of $\tilde{A}$ and is innocuous in many applications.

Let $H : \mathbb{R} \to \mathbb{R}$ be an increasing and differentiable function, and let $\Sigma^*(t) \equiv \sum_{i=1}^n x_i^*(t)$. The main comparative statics results concern how the *equilibrium aggregate* $H(\Sigma^*(t))$ and the equilibrium $x^*(t)$ vary with t.

To fix ideas and to help with the exposition of the results, consider two economic models whose equilibrium may be characterized by a system of equations like (1).

Demand for social goods. Suppose $n \geq 2$ consumers allocate income w_i between good $\mathcal{X}$ with price p_x and good $\mathcal{Y}$ with price p_y. Consumer $i's$ preferences are represented by the continuously differentiable, strictly quasiconcave utility function $u_i(x_i, y_i, x_{-i})$,where x_i and y_i are consumer $i's$ consumption levels of goods $\mathcal{X}$ and $\mathcal{Y}$, while $x_{-i} = \{x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\}$ is the vector of others' consumption of good $\mathcal{X}$. In this sense good $\mathcal{X}$ is a *social good* while good $\mathcal{Y}$ is a *private good*.[1]

Consumers solve $\max_{x_i,y_i \in \mathcal{B}_i} u_i(x_i, y_i, x_{-i})$, where $\mathcal{B}_i = \{(x_i, y_i) \geq 0 : px_i + p_y y_i \leq w_i\}$ is the set of affordable consumption bundles. If demand can be solved explicitly, denote the unique solution to this problem as

$$\begin{aligned} x_i^* &= f^i(x_{-i}; p, w_i) \text{ and} \\ y_i^* &= h^i(x_{-i}; p, w_i), \end{aligned}$$

where f^i and h^i are consumer $i's$ demand functions for goods $\mathcal{X}$ and $\mathcal{Y}$, respectively, given the price vector $p = (p_x, p_y)$, wealth and the consumption of others. Note that f^i and h^i are continuous at (p, w_i) by the theorem of the maximum.

Focus on the market for the social good. Only pure strategy equilibria exist since x_i^* is unique. Letting $w = (w_1, w_2, ..., w_n)$, an equilibrium demand system given (p, w) is defined as

$$x^* = f(x^*; p, w).$$

Since every individual's demand is continuous and constrained to $\mathcal{B}_i$, f is continuous and maps a compact and convex set into itself. Therefore, Brouwer's fixed point theorem ensures an equilibrium exists.

Formally, in this setting the vector of parameters is $t = (p, w)$. Of primary interest is how the market demand for the social good, $F(p_x) \equiv \Sigma(p_x) = \sum_{i=1}^n f^i(x^*; p_x)$ varies with price p_x when demand is differentiable at equilibrium and good $\mathcal{X}$ is not a Giffen good: $\frac{\partial f^i}{\partial p_x} \leq 0$ for all i. When is it possible for market demand to slope upwards in a stable equilibrium? The existing literature on subject, discussed in Section 7, has severely constrained how individual demand depends on others' consumption, typically through the sum of others' consumption. The analysis herein constitutes a substantial generalization.

Differentiated Products Oligopoly. Following Singh and Vives (1984), the inverse demand for firm i in a differentiated products oligopoly setting with n firms

[1] To use alternative terminology, this is a model of interdependent preferences (e.g., Pollak, 1976).

engaging in Cournot competition is

$$p_i = \alpha_i(t) - \sum_{j=1}^{n} b_{ij}(t)\, x_j, \quad \text{for } i = 1, ..., n,$$

where $\alpha_i(t) > 0$, $b_{ii}(t) > 0$, and x_j is firm $j's$ quantity. The interaction terms $(b_{ij})_{i \neq j}$ may be negative or positive, depending on whether firm j has a "business enhancing" or "business stealing" effect on firm i. Let $c_i(x_i, t)$ be firm $i's$ convex cost function. Dropping the dependence on t, each firm's profit maximizing quantity x_i^* solves

$$\alpha_i - 2b_{ii}x_i^* - \sum_{j \neq i} b_{ij}x_j - \frac{dc_i(x_i^*)}{dx_i} = 0 \ \text{ for } i = 1, ..., n. \tag{2}$$

An equilibrium $x^* = (x_1^*, ..., x_n^*)$ simultaneously solves all n of these equations.

To provide an example of a linear version of system (1), note that if $c_i(x_i) = c_i x_i$ for $c_i \geq 0$ then firm $i's$ profit maximizing quantity is

$$x_i^* = \frac{\alpha_i - c_i}{2b_{ii}} - \sum_{j \neq i} \frac{b_{ij}}{2b_{ii}} x_j \ \text{ for } i = 1, ..., n. \tag{3}$$

3 Comparative Statics with Arbitrary Interactions

3.1 Definitions and a Preliminary Result

Let $f_j^i \equiv \frac{\partial f^i}{\partial x_j}$. Then totally differentiating system (1) at equilibrium gives

$$\frac{dx_i^*}{dt} = -\frac{\partial f^i}{\partial t}\frac{1}{f_i^i} - \sum_{j \neq i} \frac{f_j^i}{f_i^i}\frac{dx_j^*}{dt}, \ i = 1, ..., n. \tag{4}$$

In a strategic context, the total effect (TE) of a parameter change on individual $i's$ action, $\frac{dx_i^*}{dt}$, can be decomposed into the *private effect* (PE) and the *interactions effect* (IE). The private effect is $PE = -\frac{\partial f^i}{\partial t}\frac{1}{f_i^i}$, since this would be player $i's$ response to a parameter change if x_{-i} were held constant. The *interaction terms* $\left(-f_j^i / f_i^i\right)_{j \neq i}$ describe how player $i's$ action changes in response to a one unit increase in player $j's$ action. Thus, the *interactions effect* for player i is simply $IE = -\sum_{j=1, j \neq i}^{n} \frac{f_j^i}{f_i^i}\frac{dx_j^*}{dt}$. The effect of a parameter change on the equilibrium aggregate $H(\Sigma^*)$ is $\frac{dH(\Sigma^*)}{dt} =$

$\frac{dH}{d\Sigma^*}\left(\sum_{i=1}^{n}\frac{dx_i^*}{dt}\right)$.[2]

To observe the effect on $\frac{dx_i^*}{dt}$ and $\frac{dH(\Sigma^*)}{dt}$ of various assumptions on the interaction terms, write system (4) in matrix-vector form,

$$\underbrace{\begin{bmatrix} 1 & f_2^1/f_1^1 & \cdots & f_n^1/f_1^1 \\ f_1^2/f_2^2 & 1 & & f_n^2/f_2^2 \\ \vdots & & \ddots & \vdots \\ f_1^n/f_n^n & f_2^n/f_n^n & \cdots & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \frac{dx_1^*}{dt} \\ \vdots \\ \\ \frac{dx_n^*}{dt} \end{bmatrix}}_{dt} = \underbrace{\begin{bmatrix} -\frac{\partial f^1}{\partial t}\frac{1}{f_1^1} \\ \vdots \\ \\ -\frac{\partial f^n}{\partial t}\frac{1}{f_n^n} \end{bmatrix}}_{\partial t}. \tag{5}$$

If A is invertible, the solution to system (5) is $dt = A^{-1}\partial t$. Note that $A = \tilde{D}\tilde{A}$, where $\tilde{D}$ is a diagonal matrix whose main diagonal is $(1/f_1^1, ..., 1/f_n^n)$. Clearly, $\det(A) \neq 0$ iff $\det(\tilde{A}) \neq 0$ since $\tilde{D} \neq 0$.

Comparative statics results depend on the type of private effects created by a parameter shock. Say that a parameter shock creates *positive* private effects if $\partial t \geq 0$ with at least one element strictly positive, $\partial t \neq 0$.[3] The vector of private effects is *uniform* if $-\frac{\partial f^i}{\partial t}\frac{1}{f_i^i} = -\frac{\partial f^j}{\partial t}\frac{1}{f_j^j}$ for all $i \neq j$. The vector of private effect is *dominant for player i* if the sum of private effects is positive and the private effect for every player other than i is less than the average: $\sum_{i=1}^{n} -\frac{\partial f^i}{\partial t}\frac{1}{f_i^i} > 0$ and $-\frac{\partial f^j}{\partial t}\frac{1}{f_j^j} < \frac{1}{n}\sum_{i=1}^{n} -\frac{\partial f^i}{\partial t}\frac{1}{f_i^i}$ for all $j \neq i$. The vector of private effects *hits only player i* if $-\frac{\partial f^i}{\partial t}\frac{1}{f_i^i} \neq 0$ but $-\frac{\partial f^j}{\partial t}\frac{1}{f_j^j} = 0$ for all $j \neq i$. The vector of private effects *misses player i* if $-\frac{\partial f^i}{\partial t}\frac{1}{f_i^i} = 0$ but $-\frac{\partial f^j}{\partial t}\frac{1}{f_j^j} \neq 0$ for some j.

Existing comparative statics results at the individual level assume the private effect hits only player i (e.g., Dixit 1986; Corchón, 1994; Acemoglu and Jensen, 2013). This is the first paper to provide results for the more general case where the private effect is dominant for player i. Results for uniform private effects are also novel.[4]

[2] To connect these ideas to the social interactions literature, say that the interactions effect *reinforces* the private effect if $sgn(PE) = sgn(IE) \neq 0$; the interactions effect *counteracts* the private effect if $sgn(PE) = -sgn(IE) \neq 0$. The *social multiplier* is $\frac{TE}{PE}$, and this is greater than one if and only if the interactions effect has the same sign as the private effect.

[3] I focus on positive private effects throughout but analogous results are available for negative private effects.

[4] Acemoglu and Jensen's (2013) concept of a "shock that hits the aggregator" can be consider a special case of uniform private effects. While not explicitly stated, Jinji's (2014) results sign the total effect for uniform private effects and effects that hit only player i.

I begin with some preliminary results which highlight the importance of the row and column sums of A^{-1} to comparative statics results. In an abuse of notation, for an invertible matrix B with inverse B^{-1} whose $(i, j)th$ element is b_{ij}^{-1}, call $\sum_{i=1}^{n} b_{ij}^{-1}$ the *jth inverse column sum of* B. Let $\mathcal{NICS}$ be the class of invertible matrices with nonnegative inverse column sums. Similarly call $\sum_{j=1}^{n} b_{ij}^{-1}$ the *ith inverse row sum of* B. Let $\mathcal{NIRS}$ be the class of invertible matrices with nonnegative inverse row sums.

Any proof not contained in the text appears in the Appendix. The notation $A(x^*)$ indicates that the matrix A should be evaluated at x^*.

Lemma 1 *Suppose a parameter shock creates positive private effects.*

(a) x_i^* *is nondecreasing if and only if the ith row of* $A^{-1}(x^*)$ *contains only nonnegative elements,*

(b) If, in addition, private effects are uniform, the equilibrium x^* *is nondecreasing if and only if* $A(x^*) \in \mathcal{NIRS}$*, and*

(c) The equilibrium aggregate $H(\Sigma^*)$ *is nondecreasing if and only if* $A(x^*) \in \mathcal{NICS}$*.*

3.2 The Main Results

In this section I show that $A \in \mathcal{NICS}$ if A is a $B-$matrix and $A \in \mathcal{NIRS}$ if A^T is a $B-$matrix, where A^T denotes the transpose of A. A *$B-$matrix* is a square matrix whose row means are positive and larger than each of the off-diagonal terms of the same row. Denote this class $\mathfrak{B}$. Precisely, the $n \times n$ matrix $\Lambda = (\lambda_{ij})$ is a $B-$matrix if, for $i = 1, ..., n$,

$$\sum_{j=1}^{n} \lambda_{ij} > 0 \quad \text{and} \tag{6}$$

$$\lambda_{ij} < \frac{1}{n} \sum_{j=1}^{n} \lambda_{ij}, \quad \forall j \neq i. \tag{7}$$

Carnicer, Goodman, and Peña (1999) show that matrices in this class have a strictly positive determinant (see Corollary 4.5). The term "$B-$matrix" is introduced in Peña (2001), but a more descriptive moniker may be *row mean positive dominant* matrices.

The result in Carnicer, Goodman, and Peña (1999) is stated for B-matrices but clearly it extends to any matrix Λ whose transpose Λ^T is a $B-$matrix since $\det(\Lambda) = \det(\Lambda^T)$.

Theorem 1

(a) If A satisfies inequalities (6)-(7), then the equilibrium aggregate $H(\Sigma^*)$ is nondecreasing for any vector of positive private effects. That is, $A(x^*) \in \mathfrak{B}$ implies $A(x^*) \in \mathcal{NICS}$.

(b) If A^T satisfies inequalities (6)-(7), then the following individual level comparative statics hold.

i. x^* is nondecreasing for uniform positive private effects.

ii. x_i^* is nondecreasing if the private effect is dominant for player i.

That is, $A^T(x^*) \in \mathfrak{B}$ implies $A(x^*) \in \mathcal{NIRS}$.

(c) If, in addition, $X = X_1 \times X_2 \cdots \times X_n$ is a rectangle and $A \in \mathfrak{B}$ for all x or $A^T \in \mathfrak{B}$ for all x then equilibrium is unique.

These results are notable for at least four reasons. First, they allow for the terms of the Jacobian to take any sign, that is, any type of interaction is allowed. Second, the individual level comparative statics apply to private effects that are dominant for player i, and this contains the set of private effects that hit only player i. Third, the hypotheses are simple to check as they involve only the entries of the Jacobian. Fourth, part (c) establishes an intimate connection between comparative statics and uniqueness.

To generate some intuition for conditions (6)-(7), it is helpful to think of system (1) as a system of best response functions in an $n-$player game on a network where the n players form the nodes. For any tuple (x, t) ,the weighted digraph describing the relation between i and j is given by the coefficient matrix

$$D = I - A.$$

The off diagonals of this matrix $\left(-f_j^i/f_i^i\right)_{i \neq j}$ is the collection of interaction terms. This matrix has zeros on the main diagonal since an increase in own action does not directly cause a further change in one's action. In this setting conditions (6)-(7) can be interpreted as different measures of a player's network centrality.

Let $\sum_{j\neq i} -f_j^i/f_i^i$ be player i's (weighted) *indegree.* Then condition (6) is satisfied if each player's indegree less than one. In other words, the positive net effect of others' actions on a player's own action cannot be to strong.

To interpret condition (7), define player *$i's$ maximal relative negative influenceability* as the maximum negative interaction effect that another player j has on player i, relative to player $i's$ indegree:

$$\max\left\{0, \max\left\{\frac{nf_j^i/f_i^i}{\left|1+\sum_{j\neq i} f_j^i/f_i^i\right|}\right\}_{j\neq i}\right\}.$$

This measure of a player's centrality captures two effects. First, it is a measure of how much a player's action decreases with an increase in other's action. This is important since a parameter shock that creates positive private effects may have the opposite sign as the private effect when there are negative interactions. Second, it roughly captures variation in negative interaction. As I illustrate in Example 1 below, this variation also plays an important role in determining whether comparative statics are well-behaved. Matrix A satisfies condition (7), and hence Theorem 1(a) applies, if each player's maximal relative negative influenceability is less than one.

In order for A^T to be a $B-$matrix so that Theorem 1(b) can be applied, each player's *outdegree*, $\sum_{j\neq i} -f_j^i/f_i^i$, and *maximal relative negative influence*,

$$\max\left\{0, \max\left\{\frac{nf_i^j/f_j^j}{\left|1+\sum_{j\neq i} f_i^j/f_i^j\right|}\right\}_{j\neq i}\right\},$$

be less than one. These measures of centrality are analogous to the indegree and maximal relative negative influenceability, but focus on how a player's action affects others' actions rather than how a player's action is affected by others' actions.

The following example of the market demand for a social good illustrates the approach and assumptions driving the results.

Example 1. Consider market demand for a social good with three consumers (players) and coefficient matrix

$$A = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ c & d & 1 \end{bmatrix}. \tag{8}$$

Think of the parameter shock as an increase in the price of the social good so that the vector of private effects is negative. Under what conditions is market demand downward sloping?

Case 1, $a = 0, b < 0, c > 0, d > 0$. In this case player 1's demand is not influenced by others' consumption. Examine (4) and observe that player $2's$ demand is increasing in player $1's$ consumption but independent of player $3's$ consumption. Player $3's$ demand is decreasing in both player 1 and player $2's$ consumption. Inverting A and solving for the slope of market demand gives

$$\frac{d\Sigma^*}{dt} = -\frac{1}{f_1^1}\frac{\partial f^1}{\partial t}(1 - b - c + bd) - \frac{1}{f_2^2}\frac{\partial f^2}{\partial t}(1 - d) - \frac{1}{f_3^3}\frac{\partial f^3}{\partial t},$$

where the coefficient on the jth private effect is the jth inverse column sum of A.

Focus on the first inverse column sum. If player 1 decreases consumption by one unit, player 2 decreases consumption by $-b$ units but player $3's$ consumption increases by c units. Moreover, player 2's decrease in consumption causes player 3 to decrease consumption by an additional $-bd$ units. Thus, $-\frac{1}{f_1^1}\frac{\partial f^1}{\partial t}(1 - b - c + bd)$ represents the contribution of player 1's private effect to the slope of market demand after other players fully respond to his change in consumption. One can interpret the remaining column sums in similar fashion.

Whether market demand is downward sloping for any vector of negative private effects clearly depends on the size of the interaction effects of players 1 and 2 on player 3, but it also depends on their variation. To see this, suppose $b = -\frac{1}{2}$ while $c = d = 0.8$. Then the first inverse column sum is 0.3 and every player's maximal relative negative influenceability is less than one. But if c and d change to 1.6 and 0, respectively, then the first inverse column sum is -0.1 even though the sum of interaction effects $c + d$ remains constant. In this specification player 3's maximal relative negative influenceability is greater than one.

Case 2. $a < 0$, $b < 0, c = d = 0$. In this case player 1 and 2's demand increases with the other's consumption but player 3's demand is independent. Intuition would suggest that market demand is downward sloping, but large feedback effects between players 1 and 2 can reverse the expected comparative statics. By direct computation we have

$$\frac{d\Sigma^*}{dt} = -\frac{1}{f_1^1}\frac{\partial f^1}{\partial t}\left(\frac{1-b}{1-ab}\right) - \frac{1}{f_2^2}\frac{\partial f^2}{\partial t}\left(\frac{1-a}{1-ab}\right) - \frac{1}{f_3^3}\frac{\partial f^3}{\partial t},$$

so that demand may slope upwards if $ab > 1$. Note that there is no negative influence, so each player's maximal relative negative influenceability is zero, but each player's indegree is less than one only if $a, b < 1$. ■

3.3 Sufficient Conditions on the Interaction Terms

While the linear conditions of Theorems 1 are simple to check, it may be useful to have conditions on the interaction terms to facilitate comparison with existing results. The special case of anonymous interaction merits attention as well. *Anonymous interaction* at x^* arises if for all i, $f_j^i = f_k^i$ for every $j, k \neq i$. This case includes aggregative games where a player's payoff depends only on some monotone function of the sum of actions (e.g., Corchón 1994).

Corollary 1 *The hypotheses of Theorem 1(a)-(b) are satisfied if either of the following conditions is satisfied at x^*.*

(a) For all i, $|f_l^i/f_i^i| < \frac{1}{2(n-1)}$ for all $j \neq i$.

(b) For all i $|f_l^i/f_i^i| < \frac{1}{n-1}$ for all $j \neq i$ and interaction is anonymous.

4 Stability

As Case 2 in Example 1 shows, the equilibrium aggregate may decrease with positive private effects even when all the individual interaction effects are nonnegative. This seems counterintuitive, even disturbing, but comparative statics analysis by way of the Implicit Function Theorem is simply a technique, and the behavioral intuition for these results is typically dynamic. If equilibrium is unstable, the dynamic and static predictions after a parameter shock differ; at stable equilibria they are the same. It is therefore interesting to evaluate whether comparative static predictions are robust to dynamic predictions for some reasonable dynamic specification. I use best reply dynamics for this purpose, and in this section I characterize stable equilibria.

The main difficulty in analyzing stability in discrete time is that standard results are available for explicitly defined systems, but system (1) may be implicitly defined. I solve this problem by transforming system (1) locally into an explicitly defined system via the Implicit Function Theorem. Once this is accomplished I prove two stability results. One is well-known but I include it for convenience and the second is kind of "folk" stability result for which I have been unable to find a reference.

Consider a deviation to x^0 "near" an equilibrium x^*. Thinking of system (1) a system of implicitly defined best response functions, one for each player, assume each player revises his action to best respond to x^0. Given the vector of actions $x(1)$ which results from this process, players again non-cooperatively revise their action to $x(2)$ to best respond to $x(1)$, and so on.

Formally, system 1 can be written as

$$\begin{aligned} f^1\left(x_1(k); x_{-1}(k-1), t\right) &= 0 \\ f^2\left(x_2\left(k\right); x_{-2}(k-1), t\right) &= 0 \\ \vdots \quad &\quad \vdots \\ f^n\left(x_n\left(k\right); x_{-n}(k-1), t\right) &= 0 \end{aligned} \tag{9}$$

where $x_{-i}\left(k-1\right) = \left(x_1\left(k-1\right), ..., x_{i-1}\left(k-1\right), x_{i+1}\left(k-1\right), ..., x_n\left(k-1\right)\right)$ with initial value $x(0) = x^0$. Intuitively, in each equation the other players' actions $x_{-i}\left(k-1\right)$ are taken as given so that $\left(x_i\left(k-1\right), t\right)$ are treated as parameters. Since $f_i^i \neq 0$, by the Implicit Function Theorem there exists a unique function z at (x^*, t) such that

$$x\left(k\right) = z\left(x\left(k-1\right), t\right). \tag{10}$$

If we consider deviations within the open neighborhood around (x^*, t) in which equation (10) is valid, we may analyze properties of z directly to determine the local stability of the equilibrium x^*

We need the following notions of stability.

Definition 1 The equilibrium point x^* of (10) is:

1. *Stable* if given $\varepsilon > 0$ there exists $\delta = \delta\left(\varepsilon\right)$ such that $\|x^0 - x^*\| < \delta$ implies $\|x(k; x^0) - x^*\| < \varepsilon$ for all $k \geq 0$, and *unstable* if it is not stable.

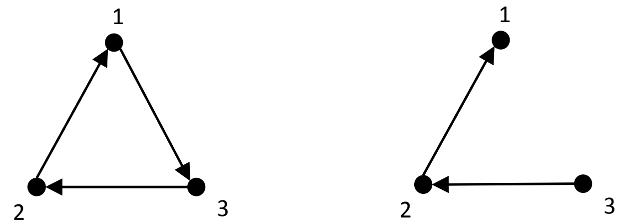

Figure 1: Interaction on a circle (left) or on a line (right).

2. *Exponentially stable* if there exists $\delta > 0$, $M > 0$, and $\eta \in (0,1)$ such that $\|x(k; x^0) - x^*\| \leq M \|x^0 - x^*\| \eta^k$, whenever $\|x^0 - x^*\| < \delta$.

By the Implicit Function Theorem, the Jacobian of system *(10)* is exactly the weighted adjacency matrix, $D = I - A$. Standard stability results for nonlinear difference equations state that equilibrium x^* is exponentially stable if $\rho(D) < 1$, where $\rho(D)$ is the spectral radius of D (e.g., Elaydi, 2000). For autonomous systems like (10), equilibrium is unstable if $\rho(D) > 1$, and may be stable or unstable if $\rho(D) = 1$.

Example 2. To motivate the following definitions and results, consider a 3-person social goods market where individual demand can be solved explicitly in terms of the others' consumption given prices and wealth. The interaction is described by the adjacency matrix

$$D = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & c \\ d & 0 & 0 \end{bmatrix} \text{ with } d \in \{0, c\} \text{ for } c > 0. \tag{11}$$

Figure 1 represents the direction of interaction depending on the value of d.

If $d = c$, then interaction takes place on a circle as in the left panel of Figure 1. Feedback effects are present since own consumption indirectly influences own demand. To see this, suppose that, by mistake, player 3 increases his consumption by one unit. In the next period, player 2 rationally responds by increasing his consumption c units while player 3 corrects his mistake and returns to equilibrium In the following period, player 1 increases his consumption c^2 units while player 2 returns to his equilibrium consumption. In the third period after the deviation, player 3 increases his consumption c^3 units while player 1 returns to equilibrium. In this way, player 3's demand is indirectly affected by own consumption through the demand response of others.

In fact, the cycle continues such that the individual whose consumption is not in equilibrium in the nth period after the deviation is c^n units away from its equilibrium value. If $c < 1$ then consumption returns to equilibrium in the limit. If $c = 1$ the equilibrium is stable to the deviation described but not exponentially stable, and if $c > 1$ equilibrium is unstable. Formally, notice that $\rho(D) = c$.

If $d = 0$, then interaction takes place on a line as in the right panel of Figure 1. Following the logic above, the economy returns to the initial equilibrium in the third period after any deviation by player 3, independent of the value of c. Formally, if $d = 0$ then D is an upper triangular matrix with zeros on the main diagonal. Since the spectrum of an triangular matrix is the same as its diagonal entries, it follows that $\rho(D) = 0$ for any c.■

Example 2 illustrates that if feedback effects are absent or limited then equilibrium is exponentially stable. This motivates the following definitions.

Definition 2 D contains a *directed cycle* if there is a sequence of interaction terms such that $-f_{i_2}^{i_1}/f_{i_1}^{i_1} \neq 0, -f_{i_3}^{i_2}/f_{i_2}^{i_2} \neq 0, ..., -f_{i_K}^{i_{K-1}}/f_{i_{K-1}}^{i_{K-1}} \neq 0$ for $i_k \in \{1, ..., K\} \subseteq \{1, ..., N\}$ and $i_1 = i_K$.

Definition 3 D is *acyclic* if it does not contain a directed cycle.

Definition 4 *Feedback effects* exist if D contains a directed cycle.

In the social goods market example above, feedback effects exist if $d = c$ since $-f_3^2/f_2^2 \neq 0, -f_2^1/f_1^1 \neq 0$, and $-f_1^3/f_3^3 \neq 0$. When feedback effects are present, a sufficient condition for stability is that the magnitude of the interaction effects is limited. I borrow the concept of "moderate social influence" from the social interactions literature to formally define this condition (e.g., Glaeser and Scheinkman, 2000; Horst and Scheinkman, 2006; Christensen and Jung, 2010). In particular, say that *actions are subject to moderate social influence at* x if $\|D\| < 1$ at x, where $\|.\|$ is any matrix norm induced by a vector norm.

Two especially useful matrix norms in this context are:

$$\begin{aligned}\|D\|_\infty &= \max_i \sum_{j=1}^{n} \left|f_j^i/f_i^i\right| \text{ (maximum absolute row sum), and} \\ \|D\|_1 &= \max_j \sum_{i=1}^{n} \left|f_j^i/f_i^i\right| \text{ (maximum absolute column sum).}\end{aligned}$$

These are functions of a player's *absolute indegree,* $\sum_{j=1}^{n} \left|f_j^i/f_i^i\right|$, and *absolute outdegree,* $\sum_{i=1}^{n} \left|f_j^i/f_i^i\right|$, respectively. When we are concerned with stability we must take the sum of absolute values since stability does not hinge on the direction of reaction, whereas comparative statics results clearly do.

If actions are subject to moderate social influence under the maximum absolute column sum norm, then there is no individual whose choices affect the actions of others too strongly. Alternatively, if actions are subject to moderate social influence under the maximum absolute row sum norm, then there are no individuals whose choices are affected too strongly by the actions decisions of others. Note that it is possible that are subject to moderate social influence under one norm but not the other. However, interactions fail to be moderate under either norm if $\left|f_j^i/f_i^i\right| > 1$ for any (i, j) pair. Notice that in Example 2, D satisfies both norms if $c < 1$ when interaction takes place on a circle.

Theorem 2 *Equilibrium x^* is exponentially stable if either of the following conditions is satisfied at equilibrium.*

(a) Feedback effects are absent.

(b) Actions are subject to moderate social influence at x^*.

Proof. (a) D is acyclic if and only if it is nilpotent. Hence, $\rho(D) = 0$.

(b) It is well known that $\rho(D) \leq \|D\|$ for any matrix norm $\|.\|$. ■

5 Comparative Statics and Stability

It is helpful to consider a 3-person example to better understand the relationship between comparative statics and stability. In this case we have

$$A = \begin{bmatrix} 1 & f_2^1/f_1^1 & f_3^1/f_1^1 \\ f_1^2/f_2^2 & 1 & f_3^2/f_2^2 \\ f_1^3/f_3^3 & f_2^3/f_3^3 & 1 \end{bmatrix}.$$

It follows from Theorems 1 and 2 that if for each player the absolute indegree, indegree, and maximal relative influenceability are moderate, the equilibrium is stable and the equilibrium aggregate is nondecreasing if the private effects are positive.

These linear conditions are simple to check. In a three person economy, the absolute indegree for each player is moderate, and hence equilibrium is stable, if

$$\left|f_j^i/f_i^i\right| + \left|f_k^i/f_i^i\right| < 1 \text{ for all } i \text{ and } j \neq k \neq i. \tag{12}$$

Figure 2 illustrates this condition for a single individual i. The vertical axis represents the effect on $i's$ action of a change in $j's$ action, while the horizontal axis represents the effect on i's action of a change in $k's$ action. The diamond in Figure 2 with vertices at $(0,1)$, $(1,0)$, $(0,-1)$, and $(-1,0)$ is the set of interaction terms that satisfy condition (12) for a single individual. If the interaction terms for all individuals lie within this set then equilibrium is exponentially stable. If there is at least one individual for whom the interaction terms are outside of this set then equilibrium is possibly unstable.

Turning to the comparative statics, the equilibrium aggregate is nondecreasing for positive private effects if each player's indegree and maximal relative negative influenceability is less than one. For player i this requires

$$\begin{aligned} 1 - f_k^i/f_i^i - f_j^i/f_i^i &> 0, & (13)\\ -f_j^i/f_i^i &< \frac{1}{3}\left(1 - f_j^i/f_i^i - f_k^i/f_i^i\right), \text{ and} & (14)\\ -f_k^i/f_i^i &< \frac{1}{3}\left(1 - f_j^i/f_i^i - f_k^i/f_i^i\right). & (15)\end{aligned}$$

The first inequality limits player $i's$ indegree while last two limit his relative negative influenceability. The triangle in Figure 2 with points at $(0,1)$, $(1,0)$, and $(-1,-1)$

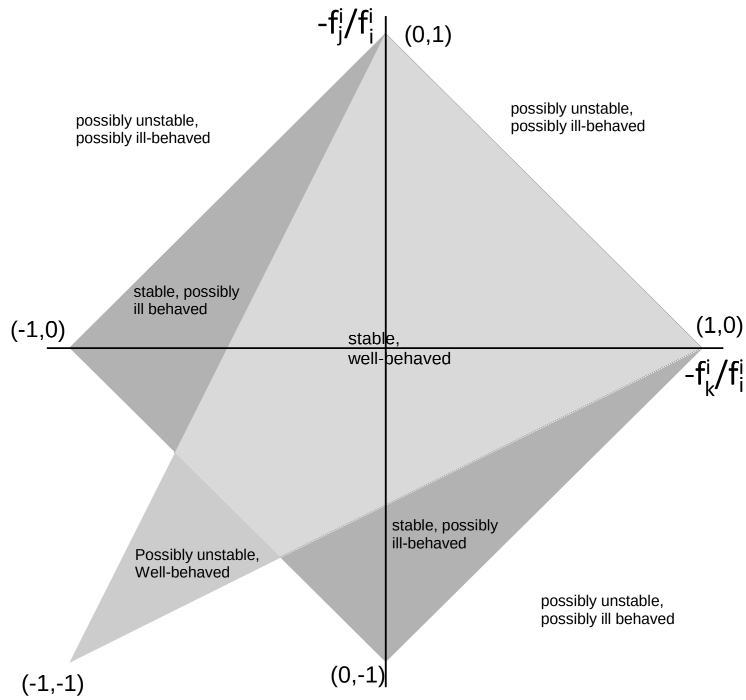

Figure 2: The Relationship Between Stability and Comparative Statics in a 3-equation system. The regions indicate the areas where equilibrium is exponenetial stable or possibly unstable, and where comparative statics are well-behaved or possibly ill-behaved.

illustrates the set of interaction terms that satisfy inequalities (13)-(15). If the interaction terms for all individuals lie within this set then the equilibrium aggregate is nondecreasing. If there is at least one individual for whom the interaction terms are outside of this set the equilibrium aggregate may decrease with positive private effects.

5.1 Strategic Complements

By *strategic complements* I mean that the interaction terms are all nonnegative, $-f_j^i/f_i^i \geq 0$ for $i \neq j, i = 1, ..., n$. As was demonstrated in Case 2 of Example 1, even the equilibrium aggregate may decrease when private effects are positive. In this section, I show that this prediction is not robust under best reply dynamics, a result suggested examining the upper right quadrant of Figure 2. In fact, I demonstrate the much stronger result that equilibrium is nondecreasing if and only if equilibrium is exponentially stable.

This result is an Implicit Function Theorem-based version of the lattice-based result in Echenique (2002) in which it is shown that if equilibrium is not monotone increasing in an exogenous parameter then equilibrium must be unstable under a broad class of adaptive dynamics.[5] My result applies to best reply dynamics, but the payoff is an incredibly simple proof of this powerful result. In addition, this approach allows for the new result that equilibrium is unique if $\rho(D) < 1$ for all x.

The proof follows from the theory of $M-$matrices (e.g., see Bapat and Raghavan, 1997). The $n \times n$ matrix Λ is called an *M-matrix* if it can be written $\Lambda = \tau I - Y$ for some nonnegative matrix Y and scalar $\tau > \rho(Y)$. One of the many interesting properties of these matrices is that $\Lambda^{-1} \geq 0$ if and only if Λ is an $M-$matrix. This fact has been known in the economics literature as early as Debreu and Herstein (1953).

Theorem 3 *Suppose there are strategic complements.*

(a) x^* is nondecreasing for any vector of positive private effects if and only if x^* is exponentially stable. Specifically, (i) $\rho(D(x^*)) < 1$ if and only if $A^{-1}(x^*) \geq 0$, and (ii) $A(x^*)$ is not invertible if $\rho(D(x^*)) = 1$.

[5] Both results can be viewed as a formal statement of Samuelson's (1947) "Correspondence Principle."

(b) *Equilibrium is unique if* $\rho(D) < 1$ *for all* x *and* $X = X_1 \times X_2 \cdots \times X_n$ *is a rectangle.*

In fact, if x^* is unstable, X_i for $i = 1, .., n$ is convex and compact, and z describes global dynamics, then play converges to a vector $\bar{x} \geq x^*$ ($\bar{x} \neq x^*$) after a parameter shock which creates positive private effects. This follows from the monotone convergence theorem since dynamic system (10) generates a monotone increasing sequence $\{x^k\}_0^\infty$ after a parameter shock. In fact, the same result applies under any adaptive dynamic where an individual increases his action if others increase theirs, whether or not players are best responding. In this sense, the comparative statics technique fails to give a satisfactory prediction of behavior at unstable equilibria. I give a simple graphical example of this result in the context of the market demand for social goods in Section 7.

Suppose z is continuous in addition to describing global dynamics. Then $\bar{x}$ is an equilibrium if x^* is an unstable interior equilibrium:

$$\bar{x} = \lim_{k \to \infty} x^k = \lim_{k \to \infty} z\left(x^k\right) = z(\lim_{k \to \infty} x^k) = z(\bar{x}).$$

A similar argument establishes the existence of a second equilibrium $\tilde{x} \leq x^*$ ($\tilde{x} \neq x^*$) to which $\{x^k\}_0^\infty$ converges if the parameter shock causes a negative private effect. In other words, the existence of an unstable interior equilibrium implies the existence of a lower and higher equilibrium.

5.2 Strategic Substitutes

By *strategic substitutes* I mean that all the interaction terms are nonpositive, or $-f_j^i/f_i^i \leq 0$ for $i \neq j$, $i = 1, ..., n$. Unsurprisingly, no result analogous to Theorem 3 is available in general. For example, suppose

$$A = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}.$$

Then $\rho(D) \leq 1$ if and only if $ab \leq 1$. But letting $e^T = (1, 1)$ we have $A^{-1}e = \frac{1}{1-ab}(1-a, 1-b)^T$. Thus, even in a stable equilibrium, a player's equilibrium action can decrease in response to parameter shock that creates a positive private effect. For example, consider $a > 1$ and $b < \frac{1}{a}$.

As suggested by the lower left quadrant of Figure 2, the results in this section demonstrate that well-behaved comparative statics are more likely to arise when interaction effects are limited in size and variation. The intuition behind limiting the size of negative interaction effects is obvious, so let me focus on the intuition for why heterogenous interaction terms can lead to a decrease in the equilibrium aggregate. Note that this situation is represented in the lower left quadrant of Figure 2 by the area formed by the two triangles inside the stability line but outside the cone of well-behaved comparative statics.

To take an example from the demand for social goods, consider recreational activities subject to congestion like downhill skiing or surfing. In each instance, the location, a ski resort or a beach that generates reliable waves, is fixed in the short run. Assume the marginal utility of the activity for any skier or surfer is decreasing with congestion. Additional participants means one is more likely to be in a collision, to have more difficulty in skiing a clean run or catching a good wave, and the interval between runs or rides is longer because of congestion at the lift or line-up. However, skilled participants may generate smaller externalities since these participants are more knowledgeable of etiquette and less likely to interfere with one's enjoyment of the activity. If willingness to pay for the activity is positively related with skill, which is not an unreasonable assumption since skilled participants likely obtained their skill from repetition, then skilled participants may be willing to pay more to be among a greater quantity of skilled participants rather than a lesser quantity of unskilled participants. Thus, a ski resort may be able to charge a higher price and attract more skiers if it can select for more highly skilled skiers. This may explain why resorts with more difficult trails are higher priced and more crowded than equally sized, nearby resorts with easier trails.

This example does not apply well to all congestion situations. Probably the most reasonable assumption for traffic congestion is anonymous effects since in the vast majority of cases each additional vehicle creates same negative externality. In this case the demand curve is downward sloping. That being said, it may be possible for toll operators to select for better drivers, and consequently face a less elastic demand curve, by selling passes only to those who have good driving records.

Turning to the results, note that under strategic substitutes each player's indegree and outdegree is negative. Thus the following corollary gives sufficient conditions under which the maximal relative negative influence and influenceability of each player is moderate.

Corollary 2 *Suppose there are strategic substitutes.*

(a) The hypotheses of Theorem 1(a) are satisfied if for all i, $\max_{j\neq i}\left(f_j^i/f_i^i\right) < \frac{1}{n-1} + \frac{n-2}{n-1}\min_{j\neq i}\left(f_j^i/f_i^i\right)$.

(b) The hypotheses of Theorem 1(b) are satisfied if for all j $\max_{i\neq j}\left(f_j^i/f_i^i\right) < \frac{1}{n-1} + \frac{n-2}{n-1}\min_{j\neq i}\left(f_j^i/f_i^i\right)$.

Stronger results are available when interaction is anonymous, but they require some additional background. Say that $A \geq 0$ is an *inverse* $M-$*matrix* if $A^{-1} = \tau I - Y$ for some nonnegative matrix Y and scalar $\tau > 0$ such that $\rho(Y) < 1$. Importantly, inverse $M-$matrices have an inverse whose main diagonal is nonnegative and nonpositive off-diagonal terms. Fully characterizing this class of matrices is an open problem, but some results exist (see Johnson and Smith, 2011). In this paper I will use results from Martínez, Michon, and Martín (1994) and Johnson and Smith (2007).

Martínez, Michon, and Martín (1994) show that strictly ultrametric matrices are inverse $M-$matrices whose inverse is strictly row and column diagonally dominant. The matrix A is a *strictly ultrametric matrix* if:

(i) A is symmetric with nonnegative entries,

(ii) $f_j^i/f_i^i \geq \min\left\{f_k^i/f_i^i, f_j^k/f_k^k\right\}$ for all $i, j, k \in \{1, ..., n\}$,

(iii) $1 > \max\left\{f_k^i/f_i^i : k \in \{1, ..., n\} \backslash i\right\}$ for all $i \in \{1, ..., n\}$.

Johnson and Smith's (2007) characterization requires A to satisfy the *strict path product property:*

$$\frac{f_j^i}{f_i^i}\frac{f_k^j}{f_j^j} \leq -\frac{f_k^i}{f_i^i}$$

for all distinct indices i, j, k such that $1 \leq i, j, k, \leq n$ with strict inequality whenever $i = k$. (Notice that $f_j^i = 0$ implies $f_k^j f_j^k = 0$ for all k.) Intuitively, if player k has an interactions effect on player i, then this effect is greater than any effect which emanates from player k through a third player. Johnson and Smith's Theorem 3 implies that if A satisfies the strict path property, $n \geq 3$, and for $i \neq j$

$$\sum_{k\neq i,j}\frac{f_j^i}{f_i^i}\frac{f_k^j}{f_j^j} \leq 1,$$

then A is an inverse $M-$matrix.

Theorem 4 *Suppose there are strategic substitutes, $n \geq 3$, and interactions are anonymous at x^*, $f^i_j / f^i_i = c_i$ for all i.*

(a) If $c_i < 1$, the equilibrium aggregate $H(\Sigma^*)$ increases with positive private effects.

(b) If $c_i < \frac{1}{\sqrt{n-2}}$ for all i, x^*_i increases if a positive parameter shock hits only player i and x^*_i decreases if a positive parameter shock misses player i.

(c) If $c_i = c < 1$ for all i, x^* increases under a uniform and positive private effects, x^*_i increases if the private effect is dominant for player i, and x^*_i decreases if a positive parameter shock misses i.

(d) If any of the hypotheses from parts (a)-(c) apply at all x and $X = X_1 \times X_2 \cdots \times X_n$ is a rectangle, then equilibrium is unique.

Proof. (a) This is a special case of Corollary 2. (b) A is an inverse $M-$matrix by Theorem 3 in Johnson and Smith (2007). (c) A is a strictly ultrametric matrix. (d) This follows immediately from the proof of Theorem 3(c) and the fact that any inverse $M-$matrix is a $P-$matrix (Horn and Johnson, 2001). ■

Notice that the individual level comparative statics apply under weaker conditions when greater homogeneity is imposed. Starting with arbitrary interactions, then anonymous interaction *or* strategic substitutes, then strategic substitutes *and* anonymous interaction, and finally strategic substitutes and identical interaction effects, the sufficient conditions on the interaction terms for well-behaved comparative statics are, for $n \geq 3$, $\frac{1}{2(n-1)} < \frac{1}{n-1} < \frac{1}{\sqrt{n-2}} \leq 1$.[6]

These results considerably generalize Dixit (1986) who considers an environment with strategic substitutes and anonymous interaction. In addition to some other conditions, Dixit shows that strict row diagonal dominance of A is sufficient for x^*_i to increase with positive private effects that hits only player i. Recall that strict row diagonal dominance requires that the diagonal entry of each row be strictly greater than the absolute sum of its off diagonal entries. In other words $\|D\|_1 < 1$. With anonymous interaction effects, Dixit's sufficient condition becomes $c_i < \frac{1}{1-n}$ for all i.

My results show that this condition is sufficient in an environment with either anonymous interaction or strategic substitutes, and in both cases apply to positive

[6]In order, see Corollary 1(a), Corollary 1(b), Corollary 2, Theorem 4(b) and Theorem 4(c).

private effects that are dominant for player i. The fact that anonymous interaction is unnecessary is especially notable. In justifying the product homogeneity assumption, Dixit writes (p. 119) that under heterogeneity, "in each row the off-diagonal elements would all be different, and the matrix would be just the general n-by-n matrix. No structure could be imposed on its inverse, and no meaningful results could emerge." Moreover, under the same hypotheses and parameter shocks as Dixit I am able to relax his condition to $c_i < \frac{1}{\sqrt{n-2}}$ for all i (Theorem 4(b)). Moreover, in contrast to the $\frac{1}{1-n}$ condition, Theorem 4(c) covers the textbook case of homogeneous product oligopoly where $b_{ij} = b$ for all $i, j \in \{1, ..., n\}$ in equation (2).

The final result of this section generalizes Dixit (1986) in a different direction by showing that strict diagonal dominance yields well-behaved comparative statics even without anonymous interaction. Loosely speaking, the equilibrium aggregate increases with positive private effect if there is no one person who "spoils the fun for everyone" in that an increase in their action causes a cumulatively large negative effect on others actions. Thus, such a "spoiler" is a necessary condition for ill-behaved comparative statics. Moreover, well-behaved individual level comparative statics arise as long as no one is too much of a "snob" whose action is sufficiently negatively influenced by a one unit increase in all others' actions.

Interestingly, the proof of this result relies only on simple algebra and the well-known Neumann expansion. This illustrates one of the benefits of studying stability in discrete time versus continuous time as in Dixit (1986).

Theorem 5 *Suppose there are strategic substitutes at x^*.*

(a) If $\|D(x^*)\|_1 < 1$, then equilibrium is exponentially stable and the equilibrium aggregate is nondecreasing with positive private effects.

(b) If $\|D(x^*)\|_\infty < 1$, then equilibrium is exponentially stable and (i) x^* is nondecreasing for uniform positive private effects, and (ii) x_i^* is nondecreasing for a positive private effect that hits only player i.

(c) If for all i, X_i is convex, f^i is monotone in x_i, and $\|D\|_1 < 1$ for all x or $\|D\|_\infty < 1$ for all x then equilibrium is unique.

6 Linear Systems

In this section I briefly investigate the case where system (1) is linear. That is, we can write

$$x_i = \alpha_i(t) - \sum_{i \neq j} \beta_{ij}(t) x_j \text{ for } \alpha_i > 0 \text{ and } i = 1, ..., n.$$

In this setting, $f_j^i / f_i^i = \beta_{ij}$. Several models fall into this class including the Cournot differentiated oligopoly with linear costs model from equation (3), and the linear quadratic payoff functions studied in Ballester, Calvó, and Zenou (2006). Bramoullé, Kranton, and D'Amours (2014) (hereafter *BKD*) study the case where $\alpha_i(t) = 1$ for all i, $\beta_{ij}(t) = \sigma g_{ij}$ for g_{ij} $\{0,1\}$ and $\sigma > 0$, and $g_{ij} = g_{ji}$.

BKD show that the sum of equilibrium actions is increasing in σ in any stable equilibrium. They do not provide results for the individual level and their proof relies on maximizing behavior as well as on the potential game structure. Theorem 5 provides similar comparative statics results on the equilibrium aggregate *and* individual level actions without such assumptions.

One can also use Theorem 5 to characterize comparative statics in the *BKD* framework in terms of a player's degree, where the *degree* $d(i)$ of player i is the number of links the player has in a network, or $d(i) = \sum_j g_{ij}$. *BKD* restrict actions to be nonnegative, and a player is *active* if his equilibrium action is strictly positive, and *inactive* if his equilibrium action equals zero. Let $d_A(j) = \sum_i g_{ij}^A$ be the degree of player j to active agents, where $g_{ij}^A = 0$ if player i is inactive; if player i is active, then $g_{ij}^A = g_{ij}$.

Corollary 3 *Consider the BKD framework.*

(a) If $d_A(i) < 1/\sigma$ for all i, then equilibrium is exponentially stable and the equilibrium aggregate is increasing for any positive parameter shock.

(b) If $d(i) < 1/\sigma$ for all active agents, then equilibrium is exponentially stable and x^* is increasing for a uniform positive private effect.

Let $\alpha = (\alpha_1, ..., \alpha_n)^T$. Since $x^* = A^{-1}\alpha$ and $dt = A^{-1}\partial t$, the linear environment allows us to establish relationships between equilibrium properties and comparative statics.

Theorem 6 *Suppose system (1) is linear and $\alpha_i = \bar{\alpha} > 0$ for all i. Then $sgn(x_i^*) = sgn(\frac{dx_i^*}{dt})$ if $\partial t > 0$ is uniform, and $sgn(\sum x_i^*) = sgn(\frac{d}{dt}H(\sum x_i^*))$ if $\partial t > 0$ is uniform.*

Proof. The result follows from the fact that $x_i^* = \left(\sum_{j=1}^{n} a_{1j}^{-1}\right)\alpha$ and $dt_i = \left(\sum_{j=1}^{n} a_{1j}^{-1}\right)\partial t$. ■

Let us apply this result to the *BKD* framework. A *degree uniform network* is a network where each player has the same degree, $d(i) = d$ for all i.[7] Theorem 6 implies that for any positive symmetric equilibrium[8] in a degree uniform network, x^* is decreasing in σ.

Corollary 4 *Consider the BKD framework for a degree uniform network. Then for any positive, symmetric equilibrium, x^* is decreasing in σ.*

Proof. $\frac{dx_i^*}{d\sigma} + \sum_{j\neq i} g_{ij}\frac{dx_j^*}{d\sigma} = -\sum_{j\neq i} g_{ij}x_j^* = -d(i)\,x_k^* = -dx_k^*$ for all i and any k, where the penultimate equality follows from symmetry and the last equality follows from the fact that the network is degree uniform. Therefore, an increase in σ represents a uniform negative shock, and the result follows from a minor modification of the proof of Theorem 6. ■

To conclude this section, I provide a cautionary example of a stable, symmetric equilibrium where the unique solution is decreasing with a uniform positive private effects. From a technical point of view, the example demonstrates the need for the assumption $\alpha_i = \bar{\alpha}$ for all i in Theorem 6.

Example 3. Consider the differentiated products oligopoly environment from Section 2. The linear case fits into the *BKD* framework and is a degree uniform network. Suppose there are 3 firms with $c_i(x_i) = 20x_i$ for all i. The inverse demand functions are

$$\begin{aligned} p_1 &= 300 - x_1 - .4x_2 - .4x_3 \\ p_2 &= 560 - 3x_1 - x_2 - .4x_3 \\ p_3 &= 560 - 3x_1 - .4x_2 - x_3 \end{aligned}$$

[7] One important example of a degree unifrom network is a complete bipartite graph where the set of nodes is partitioned into two subsets of equal size. Another example is interaction on a circle where each agent interacts with his nearest $d/2$ neighbors. In fact any static game where each player interacts with every other player may be thought of as a degree uniform network.

[8] That is, $x_i^* = x_j^* > 0$ for all i, j.

The unique equilibrium is $(x_1^*, x_2^*, x_3^*) = (100, 100, 100)$ and the corresponding prices are $(p_1^*, p_2^*, p_3^*) = (120, 120, 120)$ Constructing matrix D from the system (3) we have

$$D = \begin{bmatrix} 0 & -.2 & -.2 \\ -1.5 & 0 & -.2 \\ -1.5 & -.2 & 0 \end{bmatrix}.$$

Equilibrium is stable since $\rho(D) \approx 0.85$.

In an effort to stimulate the industry, suppose the government provides a uniform subsidy of $s = 20$ per unit for each firm so that the marginal cost of each firm net of the subsidy is zero. The new equilibrium is $(x_1^*, x_2^*, x_3^*) = \left(113\frac{1}{3}, 91\frac{2}{3}, 91\frac{2}{3}\right)$. Not only does the output of firms 2 and 3 decrease with the subsidy, but industry output also falls from 300 to $296\frac{2}{3}$.

This is somewhat surprising since the firms have identical costs and firms have the same quantities, prices, and profits in equilibrium. Without information about the underlying demand curves, the initial equilibrium is indistinguishable from a $3-$firm Cournot oligopoly with homogeneous products. Hence, one might expect that a uniform subsidy would increase the output of all firms.

The intuition is as follows. A subsidy directly increases the profitability of the next unit by lowering production costs. This is the private effect of the subsidy. An increase in a firm's output, however, has a business stealing effect on other firms in the industry, which is the interactions effect of the subsidy.

When products are homogeneous, the business stealing effect is symmetric so that each firm's output increases. In this example, however, the business stealing effect is much stronger for firm 1. The lower production costs allow firm 1 to steal enough business from firms 2 and 3 that the latter firms' output decreases with the subsidy.

7 Application to the Demand for Social Goods

> "Almost the whole value of strawberries in March...is the fact that others cannot get them." *Henry Cunynghame, 1892*

> "When a royal personage condemns a barbarous fashion, the osprey yields to artificial flowers." *A.C. Pigou, 1913*

Classical demand theory assumes preferences are independent. However, ever since Marshall's *Principles* in 1890, economists have recognized that others's consumption can influence own demand in important ways.[9] The formal literature on the shape of market demand in this case has benefitted from the development of game theory, but systematic analysis of market demand with interdependent preferences remains scarce[10]. Basic results are repeated and often rely on strong assumptions on the nature of the interdependence. In this section I use the insights from the paper to unify and clarify some results from the literature.

Leibenstein (1950) may be considered one of the first formal analyses of social markets. He considered the case of strategic complements (i.e., bandwagon effects) and strategic substitutes (i.e., snob effects). While Leibenstein argued that market demand should be downward sloping even when these effects exist, it is well known that demand in social markets may be upward sloping (e.g., Katz and Speigel, 1996; Rohlfs, 1974; Becker, 1991; Becker and Murphy, 2000). The typical intuition is simple. In the case of strategic complements, at a given price, some consumers may choose not to consume a good until it attracts a critical mass of consumers. This leads to multiple equilibria which necessitates upward sloping demand if demand is continuous and not horizontal. In the case of strategic substitutes, some individuals refuse to consume a good unless prices are high, because only then does the product become exclusive as other consumers drop out of the market.

7.1 Strategic Complements

The most abundant models of interdependent preferences deal with the case of strategic complements. The demand curves which appear in Leibenstein (1950), Rohlfs (1974) and Becker (1991) are special cases of this model. The difference between them lies in the degree to which social interactions influence individual demand. To be specific, when there are strategic complements call good $\mathcal{X}$ a *purely social good at* (p, w) *for individual* i if $f^i(0) = 0$, $f^i_j(0) = 0$ for all j, and there exists $x \geq 0$ $(x \neq 0)$ such that $f^i(x) > 0$. Purely social goods do not carry enough private value for an individual to consume unless there is a sufficient amount of consumption already taking place. This is often described as a good that requires critical mass to have value. The classic example from Rohlfs is the telephone, but many other goods fall into this

[9] See, for example, Cunynghame (1892), Veblen (1899), Pigou (1913), and Duesenberry (1949), and Morgenstern (1948).

[10] But see Pollak (1976).

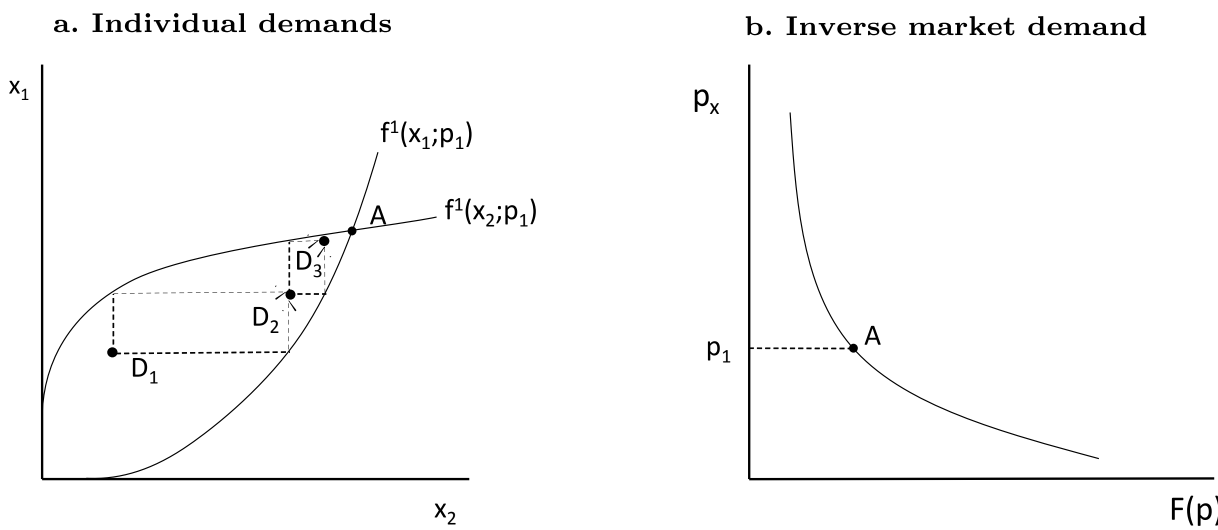

Figure 3: Strategic complements with a quasi social good.

category like social media and Internet dating sites.

Call $\mathcal{X}$ a *quasi social good at* (p, w) *for individual* i if $f^i(0) > 0$ and there exists $x \geq 0$, $(x \neq 0)$ such that $f^i(x) > f^i(0)$. Quasi social goods hold value for consumers even when no one else buys the good, but demand still increases with others' consumption. This may describe bandwagon effects or the "keeping up with the Jones'" phenomenon. Examples include computer operating systems, smart phones, cars, homes, and clothing.

Consider a market with two people whose preferences are represented by the Cobb-Douglas utility function

$$u_i = x_i^{\alpha_i + \lambda_i x_j} y_i^{\beta_i},$$

where $\lambda_i > 0$, $\beta_i > 0$, and $\alpha_i + \beta_i > 0$ for $i = 1, 2$. The parameter λ_i can be interpreted as individual i's *taste for conformity* since higher values imply a greater relative importance of x_i in overall utility. Individual $i's$ demand for good $\mathcal{X}$ is

$$f^i(x_j, p, w_i) = \begin{cases} 0 & \text{if } \alpha_i < -\lambda_i x_j \\ \frac{\alpha_i + \lambda_i x_j}{\alpha_i + \beta_i + \lambda x_j} \frac{w_i}{p_x} & \text{if } \alpha_i \geq -\lambda_i x_j \end{cases}.$$

Case 1 $\alpha_1 = \alpha_2 = 0$.

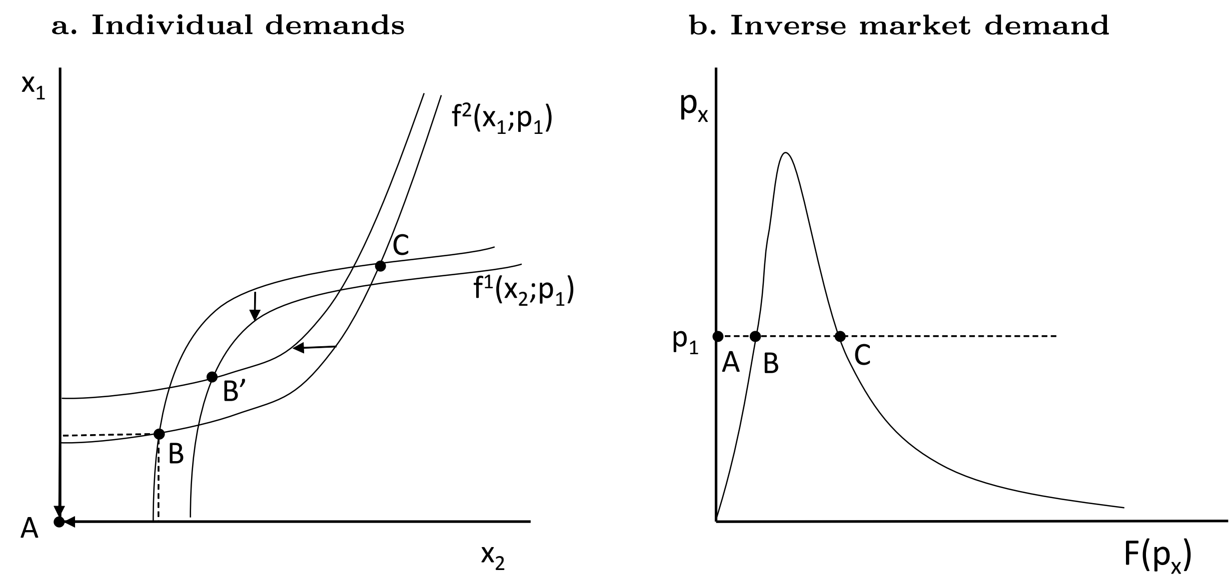

Figure 4: Strategic complements with a purely social good.

In this case, good $\mathcal{X}$ is on the margin of the class of quasi social goods. The zero equilibrium exists, but there is also a unique positive equilibrium which is easy to obtain:

$$x_i^* = \frac{\lambda_1 \lambda_2 w_1 w_2 - p_x^2 \beta_1 \beta_2}{\lambda_j p_x \left(\lambda_i w_j + p_x \beta_i\right)} \quad \text{for } i = 1, 2.$$

One can verify that x_i^* is decreasing in price for $i = 1, 2$, which implies x^* is an exponentially stable equilibrium by Theorem 3. In addition, demand is increasing in own and others' wealth, and increasing in λ_1 and λ_2.

Case 2 $\alpha_1, \alpha_2 > 0$.

Good $\mathcal{X}$ is a quasi social good for both individuals. There exists a unique, exponentially stable equilibrium as illustrated in Figure 3; best reply dynamics from a deviation to D_1 lead back to A. Moreover, equilibrium has the same comparative statics as Case 1. To see this, note that player $1's$ demand curve shifts down when p_x increases, or w_1 or λ_1 decrease. Similarly, player 2's demand curve shifts to the left when p_x decreases, or w_2 or λ_2 decrease. The market demand curve is downward sloping everywhere, as illustrated in Figure 3.b. This market demand curve is consistent with Leibenstein (1950).

Case 3: $\alpha_1, \alpha_2 < 0$.

In this case good $\mathcal{X}$ is a purely social good. The zero equilibrium always exists, and there may be zero, one, or two positive equilibria. Figure 4.a illustrates the case of two positive equilibria. Note that the equilibrium B is unstable and corresponds to an upward sloping section of market demand.

Consider the effect of an increase in price on the equilibrium at B. Player 1's demand curve shifts down while player 2's shifts left. The equilibrium locally shifts to B' which reverses the expected comparative statics with respect to p, λ, and w. Gisser, et. al. (2009) have also illustrated that pathological comparative statics arise in upward sloping sections of demand with strategic complements, but this paradox is resolved by noting that B' is an unstable equilibrium. Once the price changes, we may think of B as a deviation from B', and, as illustrated, best response dynamics leads the economy to the zero equilibrium from B. In other words, an increase in price at equilibrium B causes consumption to collapse to zero. This is consistent with the discussion after Theorem 3.

In general, the market demand curve has an inverted U shape (Figure 4.b). The equilibria to the left of the demand curve's peak are unstable whereas the equilibria to the right are stable. This is the same type of demand curve which arises in Rohlfs' (1974) application.

Case 4 $\alpha_1 < 0$ and $a_2 > 0$.

In this case, good $\mathcal{X}$ is quasi social for player 2 but purely social for player 1. Market demand may be downward sloping with a single equilibrium at high prices, multiple equilibria at intermediate prices, and then a single equilibrium at low prices where demand is downward sloping. This is illustrated in Figure 5. At low prices like p_1, player 2's consumption is sufficiently high at all levels of x_1 to induce player 1 to also consume. In this case equilibrium is unique and corresponds to point A in Figure 5.

At intermediate prices like p_2, player 2's consumption when $x_1 = 0$ is not sufficient to induce player 1 to purchase the good. This is represented by point D. However, at higher levels of x_2, player 1 would consume a positive amount. Two possible equilibria are represented by points B and C. At p_2, B and D are exponentially stable, but C is unstable. At even higher prices (not pictured), only player 2 consumes the good, but it is never enough to induce player 1 to purchase the good. As noted in Becker

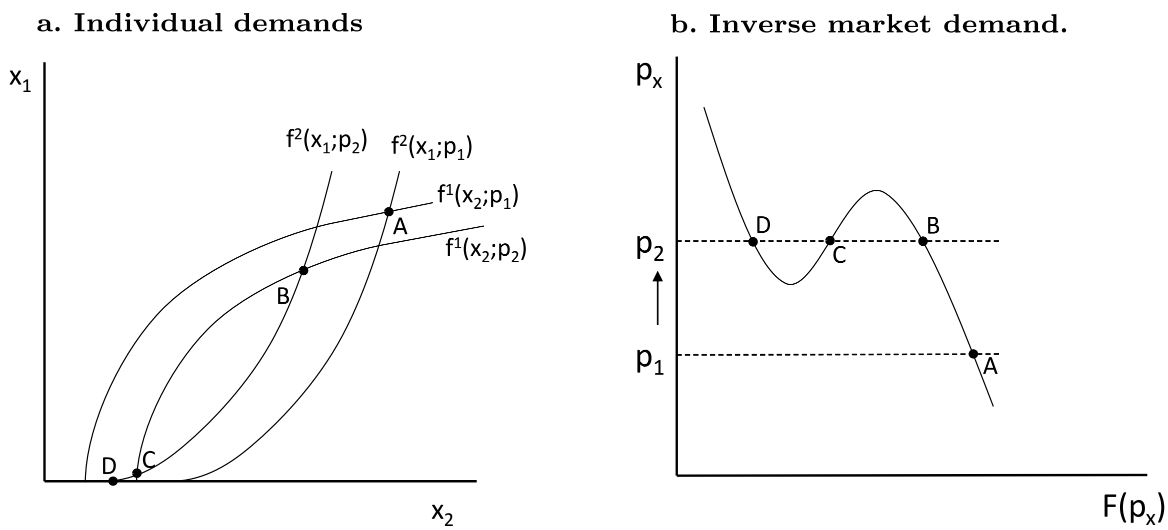

Figure 5: Good $\mathcal{X}$ is purely social for Mr. 1 but quasi social for Mr. 2.

(1991), this type of demand curve can help explain why the popularity of restaurants or bands can appear random. There are "hard core" fans like player 2 but also casual fans like player 1 who like the band only when it is popular.

These examples illustrate that when strategic complements exist, upward sloping demand should be interpreted as an increase in consumers' willingness to pay as consumption increases, rather than a situation in which sellers can increase the price and sell more. In fact, an increase in price at upward sloping points on the demand curve could cause demand to collapse. Upward sloping demand opens the possibility to a situation where there is a stable equilibrium with a higher price and quantity than an alternative equilibrium.

7.2 Strategic Substitutes

Strategic substitutes give rise to the possibility of stable, upward sloping demand. To illustrate, suppose $-f_2^1 < 0$ but $f_1^2 = 0$. Since there are no feedback effects any equilibrium is stable. Figure 6.a illustrates a situation in which player 1's demand is decreasing and convex in player 2's consumption. When price increases from p_1 to p_2, the private effect decreases consumption for both individuals: $f^2(x_1; p_2) < f^2(x_1; p_1)$ for all x_1 and $f^1(p_2; x_2) < f^1(p_1, x_2)$ for all x_2. However, the interactions effect

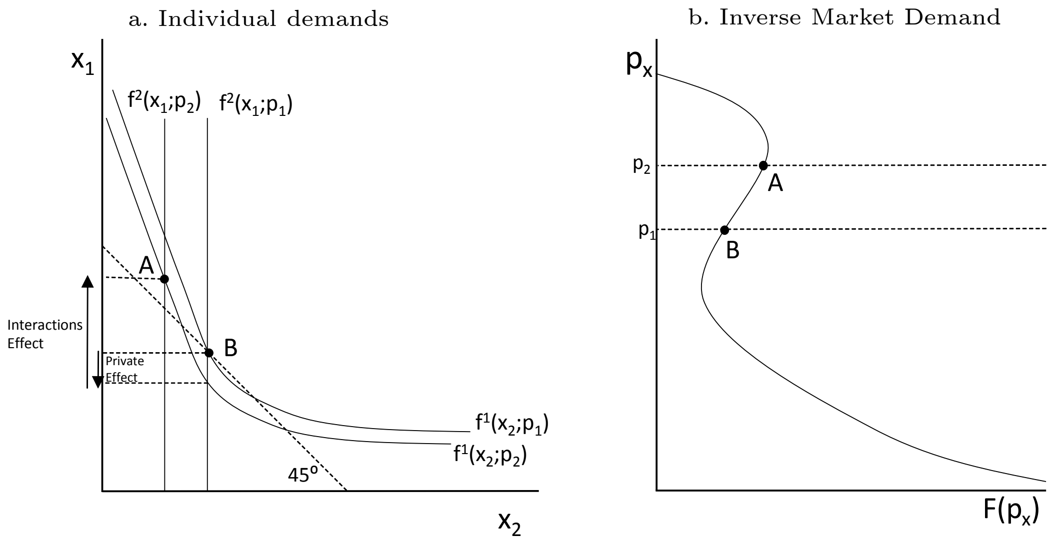

Figure 6: Stable upward sloping demand with strategic substitutes.

on player 1's consumption counteracts and in fact overwhelms the private effect: $f^1\left(p_2, f^2\left(x_1; p_2\right)\right) > f^1\left(p_1, f^2\left(x_1, p_1\right)\right)$. This effect is so strong that total quantity demanded increases. This is illustrated in panel a since B is above the 45-degree line running through A. Note that a necessary condition for market quantity to increase with price is that $-f_2^1 < -1$ for some x_2 between $f^2\left(x_1; p_1\right)$ and $f^2\left(x_1; p_2\right)$.

At prices well above p_2, the market demand curve in panel b illustrates the situation where player 2's demand falls to zero so that only player 1 is in the market. At prices sufficiently below p_1, the marginal external consumption effect of player 2 on player 1's demand is small enough so that market demand is again downward sloping.

The interesting observation here is that an increase in price in the upward sloping region of market demand will increase quantity demanded. This is consistent with the situation in which a rich snob's demand is increasing in price because higher prices force poor people out of the market towards substitutes. However, upward sloping demand can result even if the two individuals have the same wealth. For example, the "snob" may have a strong preference for the social good with a strong interactions effect while the other person only consumes a significant amount when the price is low.

8 Conclusion

This paper characterizes comparative statics of the equilibrium aggregate under for positive parameter shocks, and characterizes the comparative statics for the individual equilibrium action for uniform positive shocks as well as shocks that are dominant for an individual. Interactions are allowed to take any form. Even at this level of generality, the simple linear conditions on the interaction terms ensure that equilibrium is stable and that the equilibrium aggregate and individual equilibrium action increase with these parameter shocks. If the linear conditions apply globally, equilibrium is unique. In the context of a game on a fixed network, these linear conditions are interpreted as each player having moderate centrality in the underlying network.

An important takeaway is that in a stable equilibrium the equilibrium aggregate and individual equilibrium actions may decrease under these parameter shocks only if (i) an increase in some player's action causes at least one other player to decrease their action, and (ii) interaction effects are heterogeneous. In short, *heterogeneity matters.* If strategic substitutes exist, then there also must be one "spoiler" whose action has a strong cumulative effect on others' action for the equilibrium aggregate to decrease, and there must be a "snob" whose action strongly is influenced by others' actions for the individual level equilibrium action to decrease.

9 Appendix

Proof of Lemma 1. (a)-(b) Both facts follow from $\partial t \geq 0$ and $dt = A^{-1}\partial t$. (c) From $dt = A^{-1}\partial t$ we have for each i $\frac{df_i}{dt} = -\sum_{j=1}^{n} a_{ij}^{-1} \frac{\partial f_j}{\partial t} \frac{1}{f_j^j}$. It follows that

$$\frac{d\Sigma^*}{dt} = \sum_{i=1}^{n} \frac{dx_i^*}{dt} = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{-1} \frac{\partial f_j}{\partial t}\frac{1}{f_j^j} = -\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}^{-1}\right)\frac{\partial f_j}{\partial t}\frac{1}{f_j^j}.$$

Then $\frac{dH(\Sigma^*)}{dt} \geq 0$ for every vector $\partial t \geq 0$ if and only if $\sum_{i=1}^{n} a_{ij}^{-1} \geq 0$ for all i. ■

Proof of Theorem 1. (a) Let $\Gamma = (\gamma_{ij})$ be the cofactor matrix of A. Since $A^{-1} = \frac{\Gamma^T}{\det A}$, the inverse column sums are $\sum_{j=1}^{n} \frac{\gamma_{ij}}{\det A}$. Since $A \in \mathcal{RMPD}$ we have $\det A > 0$, and by extension $\sum_{j=1}^{n} \gamma_{ij} \geq 0$ for all i. To see this, let

$$A^i(\varepsilon) = \begin{bmatrix} 1 & f_2^1/f_1^1 & \cdots & & \cdots & f_n^1/f_1^1 \\ f_1^2/f_2^2 & 1 & & & & f_n^2/f_2^2 \\ \vdots & & \ddots & & & \vdots \\ 1-\varepsilon & 1-\varepsilon & \cdots & 1 & \cdots & 1-\varepsilon \\ \vdots & & & & \ddots & \vdots \\ f_1^n/f_n^n & f_2^n/f_n^n & \cdots & & \cdots & 1 \end{bmatrix}$$

be the matrix obtained by replacing off diagonal terms of the ith row of A with $1-\varepsilon$ for $\varepsilon \in (0,1)$. $A^i(\varepsilon) \in \mathcal{RMPD}$, so expanding along the ith row we have $\det A^i(\varepsilon) = \gamma_{ii} + (1-\varepsilon)\sum_{k\neq i}^{n} \gamma_{ik} > 0$. It follows from the continuity of the determinant function that $\lim_{\varepsilon\to 0} \det A^i(\varepsilon) = \sum_{j=1}^{n} \gamma_{ij} \geq 0$. Thus, $A \in \mathcal{NICS}$. Apply Lemma 1.c.

(b) $A^T \in \mathfrak{B}$ implies $A^T \in \mathcal{NICS}$ by part (a). It follows from $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ that $A \in \mathcal{NIRS}$. Apply Lemma 1.b.

(c) By Cramer's rule, $\frac{dx_i^*}{dt} = \frac{\det A_i}{\det A}$ where A_i is the matrix obtained from A by replacing column i with the vector ∂t. If the private effect is dominant for player i, then $A \in \mathfrak{B}$. Hence, $\det A_i > 0$.

(d) A $B-$matrix is a $P-$matrix (Peña, 2001). Thus $f : X \to \mathbb{R}^n$ is globally univalent by Theorem 4 in Gale and Nikaido (1965). ■

Proof of Corollary 1. (a) It is easy to see that the conditions imply $\|D\|_1 < 1$ and that A satisfies inequalities (6): $1 - \sum_{j\neq i} f_j^i/f_i^i > 1 - (n-1)\frac{1}{2(n-1)} > 0$. If $f_j^i/f_i^i \leq 0$, then obviously A is satisfies inequalities (7). Suppose $f_j^i/f_i^i > 0$ for some

(i,j) pair. Then

$$\max_{j\neq i} -f_j^i/f_i^i < \frac{1}{2(n-1)} = \frac{1-(n-2)\frac{1}{2(n-1)}}{n} < \frac{1-\sum_{j\neq i} f_j^i/f_i^i}{n},$$

which shows that A satisfies inequalities (7). The same argument can be made for A^T.

(b) Under anonymous interaction $\left|f_j^i/f_i^i\right| = c_i < \frac{1}{1-n}$ for all $j \neq i$. Thus, $\|D\|_\infty < 1$ and $\|D\|_1 < 1$ is equivalent to $c_i < \frac{1}{n-1}$ for all $j \neq i$. Clearly $c_i < \frac{1}{n-1}$ implies A satisfies inequalities (6). In addition, $nc_i < 1 + (n-1)c_i$ which proves that A also satisfies inequalities (7). The same argument can be made for A^T. ■

Proof of Theorem 3. (a.i) Under strategic complements, $A = I - D$ is an M-matrix if and only if $\rho(D) < 1$. The result follows from Lemmas 1.a.

(a.ii). Let $\sigma(D)$ denote the spectrum of D and $\sigma(A)$ the spectrum of A. Since the eigenvalues of D are the roots of the characteristic equation $\det(D - \lambda I) = 0$, and $A = I - D$, we have $1 - \lambda \in \sigma(A)$ if and only if $\lambda \in \sigma(D)$.

By the Perron-Frobenius Theorem, $\rho(D)$ is a real, simple eigenvalue of D if D is irreducible. Theorem 1.7.3 in Bapat and Raghavan (1997) extends this result to reducible matrices. Since $\rho(D) = 1$, A possesses a zero eigenvalue, and this implies A is not invertible.

(b) A is an $M-$matrix for all x which implies A is a $P-$matrix for all x (Horn and Johnson, 1991). Theorem 4 in Gale and Nikaido (1965) implies $f : X \to \mathbb{R}^n$ is globally univalent. ■

Proof of Corollary 2.

(a) It is obvious that the row sums of A are positive. The conditions of the corollary imply

$$\begin{aligned} nf_j^i/f_i^i &\leq n \max_{j\neq i}\left(f_j^i/f_i^i\right) \\ &< 1 + (n-2)\min_{j\neq i}\left(f_j^i/f_i^i\right) + \max_{j\neq i}\left(f_j^i/f_i^i\right) \\ &\leq 1 + \sum_{j\neq i} f_j^i/f_i^i. \end{aligned}$$

as desired.

(b) A similar argument shows that A^T satisfies inequalities (6)-(7). ■

Proof of Theorem 5.

(a) It is immediate from $\|D_f\|_1 < 1$ that equilibrium is stable and that $I + D$ has strictly positive column sums. From $A = I - D$ and the Neumann expansion we have

$$A^{-1} = (I - D)^{-1} = I + \sum_{n=1}^{\infty} (D)^n = \sum_{n=0}^{\infty} (I + D)(D)^{2n}.$$

$(D)^{2n} \geq 0$ for every n since $D \leq 0$. Lemma 2 below implies $(I + D)(D)^{2n}$ has positive column sums for every n for which $(D)^{2n} \geq 0$, and $(I + D)(D)^{2n} = 0$ for every n at which $(D)^{2n} = 0$. Positive columns sums are obviously preserved under addition, so it follows that A^{-1} has positive column sums. Therefore, market demand is downward sloping.

(b) If $\|D\|_\infty < 1$ a similar argument shows that $A \in \mathcal{NIRS}$.

(c) x^* is a fixed point of f if and only if it is a fixed point of dynamic system (9). A fixed point is unique if it is globally asymptotically stable, so the result is a consequence of Lemma 3 in Christensen and Jung (2012). ■

Lemma 2 *Let X be an $n \times n$ matrix with nonnegative column sums. Let $Y \geq 0$ be an $n \times n$ nonnegative matrix. Then $W = XY$ has nonnegative column sums. If X has at least one positive column sum and $Y > 0$, then $W = XY$ has at least one positive column sum.*

Proof. Let $X = (x_{ij})$, $Y = (y_{ij})$ and $W = (w_{ij})$. Since the column sums of X are nonnegative $\sum_{i=1}^n x_{ik} \geq 0$, and Y is nonnegative, the jth column sum of W is

$$\sum_{i=1}^{n} w_{ij} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} x_{ik} y_{kj} \right) = \sum_{k=1}^{n} \sum_{i=1}^{n} x_{ik} y_{kj} = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} x_{ik} \right) y_{kj} \geq 0.$$

The last inequality is strict if $\sum_{i=1}^n x_{ik} > 0$ and $Y > 0$. ■

Proof of Corollary 3. Order system (1) such that the first n_A rows are the best reply functions of the n_A active agents. The remaining n_I equations for the inactive agents are $x_i = 0$ in equilibrium. Then D can be partitioned as

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

where D_{11} is an $n_A \times n_A$, D_{12} is $n_A \times n_I$, D_{21} is $n_I \times n_A$ matrix and D_{22} is $n_I \times n_I$. Clearly, $D_{21} = 0$ and $D_{22} = 0$. Exponential stability in part (a) follows from the fact that the family of eigenvalues of D is the same as the union of families of eigenvalues of D_{11} and D_{22} since D is upper block triangular. The comparative statics follow from Theorem 5.

■

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