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Global Comparative Statics via an Implicit Function Theorem

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Global Comparative Statics via an Implicit Function Theorem

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Abstract

Comparative statics in smooth equilibrium models is typically characterized using the implicit function theorem, which yields local predictions based on derivatives of the equilibrium system. This paper develops a general framework for extending such local results to finite parameter changes. The analysis proceeds in two steps. First, we establish conditions under which the equilibrium system admits a globally defined, continuously differentiable selection, using either a properness-based global inversion argument or injectivity conditions applied slice-by-slice. Second, we show that global comparative statics can be obtained by integrating local responses along parameter paths. The key requirement is a cone invariance condition: parameter changes must generate shocks to the equilibrium system that lie in an admissible shock cone, and the propagation operator must map those shocks into an admissible cone of outcome changes. Under this condition, finite equilibrium changes inherit the qualitative properties of local comparative statics. A complementary result establishes that, under a strengthened finite-change hypothesis, such global behavior implies corresponding pointwise restrictions on the Jacobian. Together, these results provide a general link between local derivative-based comparative statics and global predictions in smooth equilibrium systems.

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1 Introduction

Comparative statics analyzes how equilibrium outcomes respond to changes in parameters. In smooth models, these responses are typically characterized using the implicit function theorem (IFT), which expresses local comparative statics in terms of the Jacobian of the equilibrium system. While broadly applicable, such results are inherently local, describing the effect of infinitesimal parameter changes. Extending these insights to finite changes remains a central challenge.

Several approaches address related issues from different starting points. Order-theoretic methods derive global comparative statics from monotonicity properties of the underlying mapping, without relying on differentiability or local approximations (e.g., Topkis, 2011; Vives, 1990; Milgrom and Roberts, 1990), with subsequent work extending these ideas beyond lattice environments to more general ordered structures and correspondences (e.g., Villas-Boas, 1997; Galichon, Samuelson, and Vernet, 2025; Quah and Strulovici, 2009; Che, Kim, and Kojima, 2026 and related contributions). A different line of work uses homotopy or continuation methods to compare equilibria by analyzing paths connecting them (e.g., Shiomura, 1995; Shiomura, 1998). Finally, global inversion results (e.g., Gale and Nikaido, 1965; Berry, Gandhi, and Haile, 2013) provide conditions for uniqueness of equilibrium but do not directly yield existence or comparative statics. While these approaches deliver global results under different structural assumptions, existing approaches do not provide a general framework for systematically extending derivative-based comparative statics to finite changes in a broad class of smooth equilibrium systems.

This paper develops such a framework. The central idea is that global comparative statics can be obtained by integrating local responses along parameter paths, provided that these responses satisfy appropriate invariance conditions. This yields a general method for deriving finite-change comparative statics directly from local Jacobian conditions in general smooth equilibrium systems, without imposing order structure or relying on a particular continuation procedure.

The analysis proceeds in two steps. First, we establish conditions under which the equilibrium system

$$f(y, \lambda) = 0$$

admits a globally defined, continuously differentiable selection $y = g(\lambda)$. We provide two complementary routes. The first embeds the system into the augmented mapping

$$\phi(y, \lambda) = (-f(y, \lambda), \lambda)$$

and applies a global inversion theorem. Under properness of the augmented map and nonsingularity of $D_y f$, this approach delivers existence, uniqueness, and smooth dependence of equilibrium on parameters. The second route assumes existence and uses familiar injectivity results, applied slice-by-slice to $y \mapsto f(y, \lambda)$, to establish uniqueness. Nonsingularity then allows the local IFT selections to be patched into a global C^1 equilibrium mapping.

After establishing a globally defined equilibrium mapping, we next characterize how it responds to finite parameter changes. The IFT implies that local responses are governed by

$$Dg(\lambda) = -[D_y f(g(\lambda), \lambda)]^{-1} D_\lambda f(g(\lambda), \lambda).$$

We show that global comparative statics follow by integrating these local responses along parameter paths. The key challenge is that local derivative conditions need not be preserved along parameter paths, and ensuring such invariance requires additional structure. To this end, we introduce a cone invariance condition. Admissible shocks to the equilibrium system lie in a cone S, and the linear propagation operator $-[D_y f]^{-1}$ maps S into a cone K describing admissible outcome changes. Under this condition, the effect of a finite parameter change–obtained by accumulating local responses along a path—lies in K. A complementary result establishes that, under a local finite change hypothesis, such global behavior implies corresponding pointwise restrictions on the local Jacobian. Together, these results provide a tight link between local derivative restrictions and global comparative statics.

The framework complements existing approaches. Unlike order-theoretic methods, it does not rely on monotonicity or lattice structure. Unlike homotopy-based approaches, it does not require a particular continuation procedure. And unlike global injectivity results, it connects equilibrium selection to finite-change comparative statics. Relatedly, a small literature derives global results by integrating local effects under sign invariance conditions in specific optimization settings (e.g., Suen, Silberberg, and Tseng, 2000), but does not provide a general approach for equilibrium systems of the form considered here.

There is also a large literature on global versions of the IFT. These results provide alternative conditions under which an equation system admits a globally defined solution mapping. For example, contraction-based approaches such as Zhang and Ge (2006) obtain global solvability from diagonal dominance conditions and fixed-point arguments, while topological approaches such as Sandberg (1981) establish global solvability from local solvability, compactness conditions, and an initial solution. These results are complementary to the present paper. They address the first step of the analysis: the existence and uniqueness of a global selection. To our knowledge, however, this literature does not develop the second step: conditions under which local derivative information can be aggregated along parameter paths to obtain finite-change comparative statics, or conditions under which such global behavior imposes local restrictions on the Jacobian. In this sense, the paper's main methodological contribution is the modular globalization layer. Once a smooth equilibrium selection is available, local Jacobian restrictions can be integrated along admissible paths to deliver comparative statics for finite parameter changes.

The equilibrium selection results developed below are included because they provide tractable routes to the first step that are especially natural in economic applications. The properness-based route uses the augmented mapping ϕ to convert the implicit function problem into a global inversion problem. This approach avoids the need for an initial solution, contraction arguments, or rectangular domains, and it connects directly to economically interpretable conditions such as coercivity. The injectivity-based route instead assumes existence and applies familiar square-map injectivity results slice-by-slice to $y \mapsto f(y, \lambda)$. This clarifies how tools such as Gale and Nikaido (1965), Berry, Gandhi, and Haile (2013), and related injectivity results apply to parameterized equilibrium systems. Nevertheless, the comparative statics results are modular. Any method that delivers a smooth global equilibrium mapping can be combined with the path-integration results developed here.

These results also provide a direct link between local comparative statics and the finite changes typically considered in applications. In structural and quantitative models, researchers evaluate counterfactuals involving finite parameter changes, while theoretical predictions are often derived from local derivatives. The framework developed here provides conditions under which these local predictions extend to finite changes, even in settings with multiple interacting outcomes and spillovers. In particular, it shows how qualitative predictions—such as monotonicity or relative responses across outcomes—can be established globally based on properties of the

Jacobian, rather than relying solely on numerical arguments.

We illustrate the approach in a differentiated products pricing model, where equilibrium prices solve a system of first-order conditions with rich cross-effects. The results show how standard local comparative statics—such as price responses to cost shocks—extend to finite parameter changes under familiar conditions on demand curvature and substitution patterns.

The remainder of the paper is organized as follows. Section 2 introduces the equilibrium framework. Sections 3 and 4 establish conditions for global equilibrium selection. Section 5 develops the global comparative statics results. Section 6 presents the application. Section 7 concludes.

2 Setup

Let $\mathcal{Y} \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^m$ be open sets. Endogenous outcomes are $y \in \mathcal{Y}$ and exogenous parameters are $\lambda \in \Lambda$. Let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ where

$$f(y, \lambda) = (f_1(y, \lambda), ..., f_n(y, \lambda)).$$

Given λ, an *equilibrium outcome* y satisfies the equilibrium system

$$f(y, \lambda) = 0. \tag{1}$$

Write the Jacobians with respect to outcomes and parameters as

$$D_y f(y, \lambda) \in \mathbb{R}^{n \times n} \text{ and } D_\lambda f(y, \lambda) \in \mathbb{R}^{n \times m},$$

with typical element $\partial f_i / \partial y_j$ and $\partial f_i / \partial \lambda_j$, respectively.

3 Global Equilibrium Selection via Properness

This section provides conditions under which the equilibrium system admits a unique global selection $g : \Lambda \to \mathcal{Y}$ satisfying $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$. Such a result simultaneously establishes existence and uniqueness of equilibrium for every $\lambda \in \Lambda$, and provides the smooth equilibrium mapping used in the global comparative statics analysis in Section 5. To do this, we convert the implicit function problem into an inverse function problem by augmenting the mapping f with the parameter vector.

Define the augmented mapping $\phi : \mathcal{Y} \times \Lambda \to \mathbb{R}^n \times \Lambda$ by

$$\phi(y, \lambda) = (-f(y, \lambda), \lambda).$$

The domain and codomain are subsets of the same dimensional space, $\mathbb{R}^{n+m}$. Krantz and Parks (2012) show that the inverse and implicit function theorem can be linked through such a transformation. The results below build on this insight. Specifically, we apply a global inverse function theorem to ϕ, establishing conditions under which it is a *diffeomorphism*, meaning that it is a continuously differentiable bijection (one-to-one and onto) with a continuously differentiable inverse. As claimed in the next lemma, if ϕ is a diffeomorphism, then we recover the desired equilibrium selection by evaluating ϕ^{-1} on the slice $(0, \lambda)$.

Lemma 1. *If $\phi : \mathcal{Y} \times \Lambda \to \mathbb{R}^n \times \Lambda$ is a diffeomorphism, then there exists a unique continuously differentiable mapping $g : \Lambda \to \mathcal{Y}$ such that $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$. Moreover, the Jacobian of g is given by*

$$Dg(\lambda) = -[D_y f(y, \lambda)]^{-1} D_\lambda f(y, \lambda)\big|_{y=g(\lambda)}. \tag{2}$$

Proof. Since ϕ is a bijection from $\mathcal{Y} \times \Lambda$ onto $\mathbb{R}^n \times \Lambda$, for every $\lambda \in \Lambda$ there exists a unique pair $(y, \lambda) \in \mathcal{Y} \times \Lambda$ such that $\phi(y, \lambda) = (0, \lambda)$. Define $g(\lambda) = y$. By construction, $f(g(\lambda), \lambda) = 0$, and uniqueness follows from injectivity of ϕ.

That g is C^1 follows from the fact that ϕ^{-1} is C^1. Differentiating the identity $f(g(\lambda), \lambda) = 0$ with respect to λ via the chain rule yields the stated formula for $Dg(\lambda)$. □

The next result gives necessary and sufficient conditions under which ϕ is a diffeomorphism. It applies a global inverse function theorem due to Gordon (1972). Our goal is to express the hypotheses in a form that is both strong enough to deliver a global equilibrium mapping and transparent enough to verify in economic applications.

A continuous function $F : M \to N$ is *proper* if the preimage of every compact set is compact, that is, $F^{-1}(K)$ is compact for every compact $K \subset N$.

Lemma 2. *Let $\mathcal{Y}$ and Λ be open sets, with $\mathcal{Y}$ connected and Λ simply connected, and let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ be continuously differentiable. Then ϕ is a diffeomorphism if and only if it is proper and $D_y f(y, \lambda)$ is everywhere nonsingular.*

The two conditions have distinct roles. Nonsingularity of $D_y f$ is equivalent to the nonsingularity of the Jacobian of ϕ, and is the familiar local nondegeneracy condition from the classical IFT. Properness is a global regularity condition. Economically, it rules out sequences of equilibria in which outcomes diverge while parameters remain bounded. Under the stated topological assumptions, Gordon's theorem combines these local and global conditions to promote local solvability into a global diffeomorphism.

Proof. The Jacobian matrix of ϕ has block form

$$D\phi = \begin{bmatrix} -D_y f & -D_\lambda f \\ 0 & I \end{bmatrix}.$$

Using the properties of block triangular matrices,

$$\det D\phi = \det(-D_y f) \det(I) = (-1)^n \det D_y f.$$

Hence $D\phi$ is everywhere nonsingular iff $D_y f$ is everywhere nonsingular. Moreover, because $\mathcal{Y}$ is connected and Λ is simply connected, the domain $\mathcal{Y} \times \Lambda$ is connected and the codomain $\mathbb{R}^n \times \Lambda$ is simply connected.

For the "only if" direction, suppose ϕ is a diffeomorphism. Then, by Theorem B in Gordon (1972), ϕ is proper and $D\phi$ is everywhere nonsingular. By the equivalence just established, $D_y f$ is everywhere nonsingular.

For the "if" direction, suppose ϕ is proper and that $D_y f(y, \lambda)$ is everywhere nonsingular. Then $D\phi$ is everywhere nonsingular, and Theorem B in Gordon (1972) implies that ϕ is a diffeomorphism. □

The following Theorem is an immediate corollary of Lemmas 1 and 2, and achieves the central goal of this Section.

Theorem 1. *Let $\mathcal{Y}$ and Λ be open sets, with $\mathcal{Y}$ connected and Λ simply connected, and let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ be continuously differentiable. Suppose ϕ is proper and $D_y f(y, \lambda)$ is everywhere nonsingular. Then there exists a unique continuously differentiable* *mapping $g : \Lambda \to \mathcal{Y}$ such that $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$. Moreover, the Jacobian of g is given by (2).*

3.1 Verifying Properness

The properness assumption in Theorem 1 is a global condition on the augmented mapping ϕ. This subsection provides two complementary ways to verify it through conditions on f: a primitive coercivity condition and an analytic condition on the Jacobian of f with respect to y which applies in the special case $\mathcal{Y} = \mathbb{R}^n$ and $\Lambda = \mathbb{R}^m$. Both are convenient in economic models.

3.1.1 Coercivity

First, we establish that properness of ϕ depends only on the behavior of f when parameters remain bounded. This is made precise by the following equivalence.

Lemma 3. *For any subset $C \subseteq \Lambda$, let f_C denote the restriction of f to $\mathcal{Y} \times C$. ϕ is proper iff for every compact set $C \subseteq \Lambda$, f_C is proper.*

Proof. See Appendix. □

A simple implication is that properness of f on $\mathcal{Y} \times \Lambda$ implies properness of ϕ, although the converse is false. The next result gives a coercivity condition that is useful for verification in economic models.

Proposition 1. ***(Sufficient condition for properness)*** *Suppose that for every compact set $C \subseteq \Lambda$, the following condition holds: for every sequence $\{(y^k, \lambda^k)\} \subseteq \mathcal{Y} \times C$, if $\{y^k\}$ leaves every compact subset of $\mathcal{Y}$, meaning that for every compact $B \subseteq \mathcal{Y}$ there exists N_B such that $y^k \notin B$ for all $k \geq N_B$, then*

$$\left\| f(y^k, \lambda^k) \right\| \to \infty. \tag{3}$$

Then ϕ is proper.

Proof. Let $K \subseteq \mathbb{R}^n \times \Lambda$ be a compact set. We show that $\phi^{-1}(K)$ is compact. Suppose, toward a contradiction, that $\phi^{-1}(K)$ is not contained in any compact subset of $\mathcal{Y} \times \Lambda$. Then there exists a sequence $\{(y^k, \lambda^k)\} \subseteq \phi^{-1}(K)$ with no convergent subsequence in $\mathcal{Y} \times \Lambda$.

Since $(-f(y^k, \lambda^k), \lambda^k) \in K$, the sequence $\{\lambda^k\}$ lies in the compact set $\pi_\Lambda(K) \subseteq \Lambda$. Hence failure to have a convergent subsequence in $\mathcal{Y} \times \Lambda$ must come from the y-coordinates. Thus, $\{y^k\}$ has no convergent subsequence in $\mathcal{Y}$, equivalently it leaves every compact subset of $\mathcal{Y}$. By the coercivity condition, $\left\| f(y^k, \lambda^k) \right\| \to \infty$. But $(-f(y^k, \lambda^k), \lambda^k) \in K$ and K is compact, so $\{-f(y^k, \lambda^k)\}$ is bounded, a contradiction.

Therefore, $\phi^{-1}(K)$ is contained in a compact subset of $\mathcal{Y} \times \Lambda$. Since ϕ is continuous and K is closed, $\phi^{-1}(K)$ is closed. Therefore, $\phi^{-1}(K)$ is a closed subset of a compact set, and hence compact. □

Remark 1. When $\mathcal{Y}$ is bounded, the coercivity condition reduces to divergence as $y \to \partial\mathcal{Y}$. When $\mathcal{Y}$ is unbounded, it additionally rules out escape to infinity.

A closely related boundary condition appears in general equilibrium models. Let $z(p, \omega)$ denote reduced excess demand defined on the open price simplex Δ, with ω denoting endowments. An extension of the desirability assumption in Dierker (1972) or Chichilnisky (1998) to a parameterized excess demand function is

$$\|z(p, \omega)\| \to \infty \text{ as } p \to \partial\Delta,$$

uniformly in ω on compact subsets of the endowment space. In the present framework, this condition implies that the associated augmented map

$$\phi(p, \omega) = (-z(p, \omega), \omega)$$

is proper, and therefore directly fits Proposition 1.

Applying index theory, Dierker (1972) invokes a related condition on the excess demand field to establish that the number of equilibria is odd, and in fact unique if all equilibria are locally stable. Chichilnisky (1998) instead interprets desirability as a properness condition and combines it with a nonvanishing Jacobian to obtain global invertibility and uniqueness of equilibrium.

The present approach can be viewed as a parametric extension of these ideas. Coercivity yields properness of the augmented map, and Lemma 2 then promotes local invertibility to a global diffeomorphism on open, connected domains. By Theorem 1, this delivers existence, uniqueness, and smooth dependence of equilibrium on parameters, thereby linking classical general equilibrium boundary conditions to the global comparative statics results developed below.

More generally, the coercivity condition in Proposition 1 requires that the equilibrium equations diverge whenever the endogenous variables cannot remain in a compact subset of the admissible domain, uniformly over compact parameter sets. The following examples illustrate common economic mechanisms that generate this form of coercivity.

Example 1. (Cost-Benefit Problems with Interior Solutions)

Consider a scalar choice problem in which an agent selects $y \in [\underline{y}, \bar{y}]$ to maximize $b(y, \lambda) - c(y, \lambda)$, where b and c are continuously differentiable functions. We study interior solutions on $\mathcal{Y} = (\underline{y}, \bar{y})$, where the first-order condition is

$$f(y, \lambda) = b_y(y, \lambda) - c_y(y, \lambda) = 0.$$

Suppose that for every compact set $C \subseteq \Lambda$,

$$f(y, \lambda) \to +\infty \text{ as } y \downarrow \underline{y}, \quad f(y, \lambda) \to -\infty \text{ as } y \uparrow \bar{y},$$

uniformly in $\lambda \in C$. Since $\mathcal{Y}$ is bounded, any sequence that leaves every compact subset of $\mathcal{Y}$ must approach either $\underline{y}$ or $\bar{y}$. The assumed Inada-type behavior therefore implies $|f(y, \lambda)| \to \infty$ uniformly over compact parameter sets, so Proposition 1 applies.

This is the familiar case in which marginal benefit dominates near the lower boundary and marginal cost dominates near the upper boundary, ensuring an interior optimum. The coercivity condition formalizes that the first-order condition cannot remain bounded near the boundary.

In this scalar case, nonsingularity of f_y implies that $f(\cdot, \lambda)$ is strictly monotone, so uniqueness follows immediately and existence requires only a sign change. The coercivity condition used here is stronger than necessary in one dimension, but it provides a convenient sufficient condition that extends naturally to higher dimensional systems, where local invertibility no longer guarantees global uniqueness.

Example 2. (Strategic Interaction with Strong Own Effects)

Consider an n-player game in which each player i chooses $y_i \in [\underline{y_i}, \bar{y}_i]$, and interior equilibria are characterized by the first-order conditions

$$f_i(y,\lambda) = \frac{\partial b_i(y,\lambda)}{\partial y_i} - \frac{\partial c_i(y,\lambda)}{\partial y_i}, \quad i = 1, ..., n.$$

Suppose that for every compact set $C \subseteq \Lambda$, and for each i,

$$f_i(y,\lambda) \to +\infty \text{ as } y_i \downarrow \underline{y_i}, \quad f_i(y,\lambda) \to -\infty \text{ as } y_i \uparrow \bar{y}_i,$$

uniformly over $y_{-i} \in \prod_{j \neq i}(\underline{y_j}, \bar{y}_j)$ and $\lambda \in C$.

Since $\mathcal{Y} = \prod_i(\underline{y_i}, \bar{y}_i)$ is bounded, any sequence leaving every compact subset of $\mathcal{Y}$ has at least one coordinate approaching its boundary. By assumption, the corresponding component $f_i(y,\lambda)$ diverges in magnitude, uniformly over the remaining actions and over compact parameter sets. Hence, $\|f(y,\lambda)\| \to \infty$ whenever y leaves every compact subset of $\mathcal{Y}$, and Proposition 1 applies.

Observe that this condition simultaneously excludes boundary equilibria. Combined with the assumption that the Jacobian $D_y f$ is everywhere nonsingular, Theorem 1 ensures the existence of a unique equilibrium, and that the equilibrium function $g(\lambda)$ is continuously differentiable.

This example extends the first one to show that even when preferences and costs depend on the full vector of actions, coercivity holds provided that marginal incentives in each direction become arbitrarily strong near the boundary, uniformly over the behavior of other agents.

Remark 2. Proposition 1 is stated for open domains, while economic models are often formulated on closed sets (e.g., nonnegative quantities or prices). In such cases, illustrated explicitly in these examples, one works with the interior of the economically relevant domain and verifies that the equilibrium conditions diverge as the boundary is approached. Since any sequence leaving every compact subset of a bounded open domain must approach the boundary, the sequence-based coercivity condition applies directly. The condition also rules out boundary equilibria, so the open domain formulation is without loss for the interior equilibrium analysis.

Example 3. (Log-Share Systems and Multinomial Logit Demand)

Let

$$\mathcal{Y} = \left\{ s \in \mathbb{R}^J_{++} : \sum_{j=1}^{J} s_j < 1 \right\}$$

be the interior of the simplex of market shares, where s_j is the share of product j and $s_0 = 1 - \sum_{j=1}^{J} s_j$ is the outside option. In multinomial logit demand systems, equilibrium conditions are often expressed in terms of log shares:

$$f_j(s, \lambda) = \log s_j - \log s_0 - a_j(s, \lambda), \quad j = 1, ..., J,$$

where $a_j(s, \lambda)$ captures mean utility, costs, or other structural components and is bounded on $\mathcal{Y} \times C$ for compact $C \subseteq \Lambda$.

Since $\mathcal{Y}$ is bounded, any sequence leaving every compact subset of $\mathcal{Y}$ must approach the boundary of the share simplex; equivalently, $\min\{s_0, s_1, ..., s_J\} \to 0$. Fix a compact set $C \subseteq \Lambda$, and suppose $\lambda^k \in C$. Assume, toward a contradiction, that $\left\|f(s^k, \lambda^k)\right\|$ does not diverge along such a sequence. Then along some subsequence, each

$$f_j(s^k, \lambda^k) = \log s_j^k - \log s_0^k - a_j(s^k, \lambda^k)$$

is bounded. Since the a_j's are bounded on $\mathcal{Y} \times C$, it follows that each log ratio $\log(s_j^k / s_0^k)$ is bounded along this subsequence. Hence there exist constants $0 < c < C < \infty$ such that

$$c s_0^k \le s_j^k \le C s_0^k \text{ for all } j.$$

Since $s_0^k + \sum_{j=1}^{J} s_j^k = 1$, these inequalities imply that s_0^k and all s_j^k are bounded away from zero along the subsequence. This contradicts the assumption that the sequence leaves every compact subset of $\mathcal{Y}$. Therefore, $\left\|f(s^k, \lambda^k)\right\| \to \infty$ whenever s^k leaves every compact subset of $\mathcal{Y}$, uniformly over compact parameter sets. Proposition 1 applies.

Combined with the assumption that the Jacobian $D_s f$ is everywhere nonsingular, Theorem 1 ensures the existence of a unique equilibrium and a continuously differentiable equilibrium function $g(\lambda)$.

In multinomial logit models, predicted market shares lie in the interior of the share simplex: product shares are strictly positive and the outside share is also strictly positive. The log-share transformation converts this interiority into divergence at the boundary. As a sequence approaches the boundary of the simplex, at least one log-share ratio $\log(s_j/s_0)$ must diverge in magnitude, provided the remaining structural terms are bounded. This is the same boundary behavior underlying the log-share inversion used in the identification and estimation of multinomial logit demand systems

(Berry, 1994; Berry, Levinsohn, and Pakes, 1995).

Remark 3. On unbounded domains, Proposition 1 requires an additional growth condition at infinity. For example, if $\mathcal{Y} = \mathbb{R}^n$ and $\|f(y,\lambda)\| \to \infty$ as $\|y\| \to \infty$, uniformly for λ in compact sets, then ϕ is proper. An alternative route for the $\mathcal{Y} = \mathbb{R}^n$ case is given in the next subsection.

3.1.2 Bounding the Inverse when $\mathcal{Y} = \mathbb{R}^n$ and $\Lambda = \mathbb{R}^m$.

When the equilibrium system is defined on all of $\mathbb{R}^{n+m}$, global invertibility can alternatively be verified using analytic Hadamard–Lévy conditions that bound the norm of the inverse Jacobian. More generally, for maps F between Banach spaces, uniform bounds on $\|DF(x)^{-1}\|$ promote local invertibility to a global diffeomorphism under standard hypotheses (e.g., Rheinboldt, 1969). In economic applications, such bounds can often be obtained from matrix-class restrictions on $D_y f$ such as strict diagonal dominance or B-matrices. Because these classes are variously invoked to obtain comparative statics results, this approach yields a consistent set of restrictions that can support both equilibrium selection and comparative statics.

Proposition 2. *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable and that $D_y f(y,\lambda)$ is everywhere nonsingular. Let $\phi(y,\lambda) = (-f(y,\lambda),\lambda)$. If*

$$\sup_{(y,\lambda)} \left\|[D_y f(y,\lambda)]^{-1}\right\| \leq K \text{ and } \sup_{(y,\lambda)} \|D_\lambda f(y,\lambda)\| \leq L \tag{4}$$

for some constants $K, L < \infty$, then ϕ is a diffeomorphism. Consequently, there exists a unique, continuously differentiable mapping $g : \mathbb{R}^m \to \mathbb{R}^n$ such that $f(g(\lambda),\lambda) = 0$ for all $\lambda \in \mathbb{R}^m$ whose Jacobian is given by (2).

Proof. We verify the conditions of Theorem 3.11 in Rheinboldt (1969) such that ϕ is a diffeomorphism. The map $\phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is continuously differentiable and

$$D\phi(y,\lambda) = \begin{bmatrix} -D_y f(y,\lambda) & -D_\lambda f(y,\lambda) \\ 0 & I \end{bmatrix},$$

so $\det D\phi = \det(-D_y f)$, which is nonzero everywhere by assumption.

Moreover,

$$(D\phi(y,\lambda))^{-1} = \begin{bmatrix} -(D_y f(y,\lambda))^{-1} & -(D_y f(y,\lambda))^{-1} D_\lambda f(y,\lambda) \\ 0 & I \end{bmatrix}.$$

Hence, by the triangle inequality,

$$\left\|(D\phi(y,\lambda))^{-1}\right\| \leq \left\|\begin{bmatrix} -(D_y f)^{-1} & 0 \\ 0 & I \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & -(D_y f)^{-1} D_\lambda f \\ 0 & 0 \end{bmatrix}\right\|.$$

Using submultiplicativity of the norm,

$$\sup_{(y,\lambda)} \left\|(D\phi(y,\lambda))^{-1}\right\| \leq \max\{K, \|I\|\} + KL < \infty.$$

It follows from Theorem 3.11 in Rheinboldt (1969) that ϕ is a diffeomorphism. The remaining conclusion follows from Lemma 1. □

The parameter Jacobian $D_\lambda f$ encodes the direct impact of shocks on outcomes, and $-D_y f^{-1}$ describes how they propagate through the system. In this sense, condition (4) says that the shock must be controlled and the equilibrium interactions cannot be too intense; otherwise, multiple equilibria can arise. The condition on $D_\lambda f$ is typically mild in economic applications, while the condition on $D_y f$ imposes economically meaningful structure.

Although Proposition 2 is stated on $\mathbb{R}^{n+m}$, it is often possible to reformulate economic models defined on restricted domains (e.g., positive prices or quantities) so that the equilibrium system is well defined on the entire space, and then show that any equilibrium lies in the economically relevant region. We illustrate this approach in Section 6.

We now provide conditions under which the uniform bound on $[D_y f(y,\lambda)]^{-1}$ holds. The conditions below build on matrix restrictions that have been widely used to discipline local comparative statics. In this way, they provide a bridge between local comparative statics results and the existence of a global equilibrium mapping, and will play a central role in the globalization results developed in Section 5.

Strict Diagonal Dominance with margin. Several authors impose that $-D_y f$ is strictly diagonally dominant (SDD) to obtain local comparative statics results (e.g., Dixit, 1986; Nti, 1997; Norris, Johnson, and Spitkovsky, 2023) Specifically, suppose that $-D_y f(y,\lambda)$ is SDD by rows with a uniform margin $\varepsilon > 0$, that is,

$$\left|\frac{\partial f_i}{\partial y_i}\right| \geq \sum_{j\neq i}\left|\frac{\partial f_i}{\partial y_j}\right| + \varepsilon,\ i = 1, ..., n. \tag{5}$$

Then Varah (1975) implies

$$\left\|[D_y f(y,\lambda)]^{-1}\right\|_\infty \leq \frac{1}{\varepsilon}$$

Thus, strict diagonal dominance with margin provides a simple sufficient condition for the uniform inverse bound.

B-matrices with margin. The class of B-matrices has also proven useful in disciplining comparative statics (e.g., Christensen, 2019; Christensen, 2024). Let $A = -D_y f$. Define

$$r_i^+ \equiv \max\{0, a_{ij} | j \neq i\},$$

as the largest positive off-diagonal element in row i, with $r_i^+ = 0$ if all off-diagonal elements are strictly negative. Then A is a B-matrix with margin $\varepsilon > 0$ if

$$\sum_{j=1}^n a_{ij} \geq n r_i^+ + \varepsilon,\ i = 1, ..., n. \tag{6}$$

Define

$$\beta_i \equiv \left(a_{ii} - r_i^+\right) - \sum_{j\neq i}\left|a_{ij} - r_i^+\right|$$

and $\beta \equiv \min_i\{\beta_i\}$.

By Theorem 2.2. in García-Esnaola and Peña (2009), when $\varepsilon = 0$ and (6) holds with strict inequality,

$$\left\|A^{-1}\right\|_\infty \leq \frac{n-1}{\min\{\beta, 1\}}.$$

When $\varepsilon > 0$ and (6) holds weakly, we can write

$$\left\|A^{-1}\right\|_\infty \leq \frac{n-1}{\min\{\varepsilon, 1\}}.$$

4 Global Equilibrium Selection via Injectivity

The previous section used the augmented map to establish existence, uniqueness, and smooth dependence of equilibrium. This section takes a different route. It applies familiar injectivity results slice-by-slice to establish uniqueness conditional on existence. Once each parameter value admits a unique equilibrium and $D_y f$ is nonsingular, the local IFT patches the local selections into a smooth global equilibrium mapping. The following lemma makes this patching argument explicit.

Lemma 4. *Let $\Lambda \subseteq \mathbb{R}^m$ and $\mathcal{Y} \subseteq \mathbb{R}^n$ be open sets, and let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ be C^1. Suppose that for every $\lambda \in \Lambda$, there exists a unique $y \in \mathcal{Y}$ such that $f(y, \lambda) = 0$ and that $D_y f(y, \lambda)$ is everywhere nonsingular. Then there exists a unique C^1 mapping $g : \Lambda \to \mathcal{Y}$ such that $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$, with Jacobian Dg given by (2).*

Proof. Because $D_y f(y, \lambda)$ is everywhere nonsingular, the implicit function theorem implies that for each $\lambda_0 \in \Lambda$, there exists a neighborhood U of λ_0 and a C^1 map $g_U : U \to \mathcal{Y}$ such that $f(g_U(\lambda), \lambda) = 0$ for all $\lambda \in U$. By uniqueness of equilibrium for each λ, any two such local selections must agree on overlaps. Hence these local C^1 selections patch together to define a globally well-defined C^1 map $g : \Lambda \to \mathcal{Y}$. Finally, differentiating the identity $f(g(\lambda), \lambda) = 0$ with respect to λ via the chain rule yields the stated formula for $Dg(\lambda)$. □

4.1 Gale-Nikaido Matrix Conditions

Gale and Nikaido (1965) provide global injectivity results for square maps $F : \Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^n$. These are particularly appealing in the present context because they translate P-matrix and weak positive quasi-definiteness restrictions on the Jacobian into global one-to-one behavior. Such restrictions align with familiar Jacobian restrictions that discipline local comparative statics—B-matrices, M-matrices, and SDD matrices with strictly positive diagonal are P-matrices, while smooth optimization problems naturally generate quasi-definite Jacobians.

First recall the following definitions from Gale and Nikaido (1965) and Allen (2022). An *open rectangular region* is the cross product of open intervals in $\mathbb{R}^n$. A real $n \times n$ matrix A is *P-matrix* if all principal minors are positive. It is a *weak P-matrix* if its determinant is positive and all other principal minors, of order less than

n, are nonnegative. A is *weakly positive quasi-definite* if it has a positive determinant and its symmetric part, $\frac{1}{2}(A + A')$, is positive semidefinite.

Theorem 2. *Let $\Lambda \subseteq \mathbb{R}^m$ and $\mathcal{Y} \subseteq \mathbb{R}^n$ be open sets. Let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ be C^1, and suppose that for every $\lambda \in \Lambda$, there exists some $y \in \mathcal{Y}$ such that $f(y, \lambda) = 0$. Assume further that either:*

1. *for every (y, λ), $-D_y f(y, \lambda)$ is a weak P-matrix and $\mathcal{Y}$ is an open rectangular region; or*
2. *for every (y, λ), $-D_y f(y, \lambda)$ is weakly positive quasi-definite and $\mathcal{Y}$ is open and convex.*

Then there exists a unique C^1 mapping $g : \Lambda \to \mathcal{Y}$ such that $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$, with Jacobian Dg given by (2).

Proof. Fix $\lambda \in \Lambda$ and define $F_\lambda : \mathcal{Y} \to \mathbb{R}^n$ by $F_\lambda(y) = -f(y, \lambda)$. Under assumptions (1) and (2), respectively, F_λ satisfies the hypotheses of Theorems 4w and 6w in Gale and Nikaido (1965). Hence, F_λ is injective on $\mathcal{Y}$. Therefore, for each λ, there is at most one solution to $f(y, \lambda) = 0$. Existence is assumed, so the solution is unique for each λ. Since $-D_y f(y, \lambda)$ is nonsingular under either assumption, the hypotheses of Lemma 4 are satisfied, and the result follows. □

The convexity requirement in the quasi-definite case can be weakened using Proposition 2 in Allen (2022). Suppose $\mathcal{Y} \subseteq U$, where $\mathcal{Y}$ is open and U is open and convex. If, for each λ, the slice $F_\lambda(y) = -f(y, \lambda)$ extends to a differentiable monotone map on U, and $D_y F_\lambda(y)$ is nonsingular on $\mathcal{Y}$, then Allen's result implies that F_λ is injective on $\mathcal{Y}$. Equivalently, under differentiability, monotonicity can be expressed as positive semidefiniteness of the symmetric part $\frac{1}{2}\left(D_y F_\lambda + D_y F_\lambda'\right)$ on U.

4.2 A Connected Substitutes Approach

Another noteworthy approach to injectivity follows Berry, Gandhi, and Haile (2013). Their Corollary 2 implies that, under a weak substitutes (Z-matrix) condition, a mapping $F : \Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^n$ open, is globally invertible if its Jacobian is nonsingular everywhere and is a weakly column diagonally dominant Z-matrix.[1]

[1] A *Z-matrix* is a matrix with nonpositive off-diagonal elements.

In their setting, F is a demand system in which each component represents the demand for a particular good. They interpret the condition on the Jacobian of F as a "connected substitutes" condition.

Theorem 3. *Let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ be C^1 with $\mathcal{Y} \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^m$ open. Suppose that for every $\lambda \in \Lambda$, there is some $y \in \mathcal{Y}$ such that $f(y, \lambda) = 0$. Suppose further that, for every $(y, \lambda) \in \mathcal{Y} \times \Lambda$, $-D_y f(y, \lambda)$ is nonsingular and is a weakly column diagonally dominant Z-matrix. Then there exists a unique C^1 mapping $g : \Lambda \to \mathcal{Y}$ such that $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$. Moreover, the Jacobian of g is given by (2).*

Proof. For each fixed λ, define $F_\lambda(y) = -f(y, \lambda)$. By Corollary 2 of Berry, Gandhi, and Haile (2013), F_λ is injective on $\mathcal{Y}$. Hence, for each λ, there is at most one solution to $f(y, \lambda) = 0$. Existence is assumed, so the solution is unique for each λ. The hypotheses of Lemma 4 are thus satisfied, and the result follows. □

4.3 Discussion

Gale and Nikaido (1965) and Berry, Gandhi, and Haile (2013) provide conditions under which a square mapping is injective, and hence, under nonsingularity, has a C^1 inverse on its image. In both cases, however, these results do not address surjectivity beyond the image itself, and therefore do not guarantee existence of a solution to $f(y, \lambda) = 0$ for all parameter values.

By contrast, the approach in Section 3 establishes that the augmented map $\phi(y, \lambda) = (-f(y, \lambda), \lambda)$ is a global diffeomorphism onto its codomain under a properness condition on an open and connected set, without restricting the Jacobian beyond nonsingularity. This ensures that $(0, \lambda)$ lies in the image of ϕ for every λ, yielding a globally defined C^1 equilibrium function $g : \Lambda \to \mathcal{Y}$. In other words, Theorem 1 establishes existence and uniqueness of equilibrium for all $\lambda \in \Lambda$, while Theorems 2 and 3 assume existence and establish uniqueness for all $\lambda \in \Lambda$. The trade-off is that the properness-based approach requires verifying global boundary behavior whereas the injectivity approach relies on verifying properties of the Jacobian.

Finally, Theorems 2 and 3 translate familiar injectivity results for square maps into the parameterized equilibrium setting. This translation is useful because the associated matrix conditions are familiar in economic applications and align naturally with the comparative statics restrictions used below.

5 Global Comparative Statics

Having established conditions for the existence of a global C^1 equilibrium mapping $g : \Lambda \to \mathcal{Y}$, we now turn to the paper's main contribution—characterizing how equilibrium responds to finite parameter changes. We start with sufficient conditions and then consider necessity.

It is worth emphasizing that the conditions for global equilibrium selection are independent of the comparative statics results below, allowing the latter to be applied whenever a smooth equilibrium mapping is available. In the classical IFT, local equilibrium selection and comparative statics are obtained jointly from the same Jacobian condition. By contrast, the present framework separates these steps at the global level, first establishing the existence of a smooth equilibrium mapping and then deriving comparative statics by integrating its local derivatives.

5.1 Sufficient Conditions for Globalization

When both the endogenous outcome and the parameter are scalars ($n = m = 1$), the logic of the global extension is transparent. If there exists a differentiable equilibrium function $y = g(\lambda)$ on the interval $[\lambda_0, \lambda_1]$ such that $f(g(\lambda), \lambda) = 0$, the chain rule implies

$$g'(\lambda) = -\frac{\partial f(y, \lambda)/\partial \lambda}{\partial f(y, \lambda)/\partial y}.$$

This is exactly the local IFT formula.

If the parameter perturbation positively shocks the equilibrium equation,

$$\partial f(y, \lambda)/\partial \lambda \geq 0,$$

and the propagation term satisfies $-[\partial f(y, \lambda)/\partial y]^{-1} > 0$ for all $\lambda \in [\lambda_0, \lambda_1]$, then $g'(\lambda) \geq 0$ everywhere on the interval. Integrating yields

$$g(\lambda_1) - g(\lambda_0) = \int_{\lambda_0}^{\lambda_1} g'(\lambda) d\lambda = \int_{\lambda_0}^{\lambda_1} \left(-\frac{\partial f(y, \lambda)/\partial \lambda}{\partial f(y, \lambda)/\partial y} \right) d\lambda \geq 0.$$

Thus the global comparative statics conclusion follows directly from integrating the local derivative along the equilibrium path.

This scalar case illustrates the general logic of our approach. Local comparative statics describe infinitesimal responses, and global comparative statics are obtained by integrating those responses between two parameter values. In higher dimensions, the same idea applies, but sign restrictions must be replaced by set-valued restrictions on admissible directions of change. We use cones for this purpose.[2]

In higher dimensions, the local derivative of the equilibrium mapping is

$$Dg(\lambda) = -[D_y f(g(\lambda), \lambda)]^{-1} D_\lambda f(g(\lambda), \lambda).$$

This expression separates the comparative statics into two components. First, a parameter movement induces a shock to the equilibrium equations through $D_\lambda f$. Along a path $\gamma(t)$ in parameter space, the relevant object is the directional shock $D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t)$. Second, the linear propagation operator, $-[D_y f]^{-1}$ governs how the shock transmits to equilibrium outcomes. The *shock cone* S describes admissible shocks to the system, while the *outcome change cone* K describes admissible changes in equilibrium outcomes. In the scalar case described above, $S = K = \{x \in \mathbb{R} : x \geq 0\}$.

Our main result provides conditions such that if, along a path connecting two parameter values, shocks lie in S and the propagation operator maps S into K, then the resulting equilibrium change also lies in K. This is the central globalization result of the paper.

We begin with a technical lemma establishing that integration preserves membership in a closed convex cone.

Lemma 5. *Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Let $h : [a, b] \to \mathbb{R}^n$ be Riemann integrable. If $h(t) \in K$ for all $t \in [a, b]$, then $\int_a^b h(t)dt \in K$.*

Proof. Fix a partition $P = \{a = t_0 < t_1 < \cdots t_N = b\}$ and choose $\tau_i \in [t_{i-1}, t_i]$. The corresponding Riemann sum is

$$R(P, \tau) \equiv \sum_{i=1}^{N} h(\tau_i)(t_i - t_{i-1}).$$

Because K is a convex cone, finite sums of elements of K with nonnegative coefficients remain in K. Hence, $R(P, \tau) \in K$.

[2] A *cone* is a nonempty subset of $\mathbb{R}^n$ that is closed under multiplication by nonnegative scalars, that is, $K \subset \mathbb{R}^n$ is a cone if $x \in K$ implies $\alpha x \in K$ for every $\alpha \geq 0$.

Since h is Riemann integrable, there exists a sequence of Riemann sums converging to $\int_a^b h(t)dt$. Because K is closed and each Riemann sum lies in K, the limit must also lie in K. Therefore, $\int_a^b h(t)dt \in K$. □

Theorem 4. *Let $K, S \subset \mathbb{R}^n$ be closed convex cones. Let $\mathcal{Y} \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^m$ be open sets, and let $f : \mathcal{Y} \times \Lambda \to \mathbb{R}^n$ be continuously differentiable. Suppose there is a C^1 equilibrium selection $g : \Lambda \to \mathcal{Y}$ such that*

$$f(g(\lambda), \lambda) = 0 \quad \forall \lambda \in \Lambda,$$

and suppose that $D_y f(g(\lambda), \lambda)$ is nonsingular for all $\lambda \in \Lambda$. Fix $\lambda_0, \lambda_1 \in \Lambda$ and let $\gamma : [0, 1] \to \Lambda$ be a continuously differentiable path with $\gamma(0) = \lambda_0$ and $\gamma(1) = \lambda_1$. Assume that for every $t \in [0, 1]$,

1. *Shock restriction: $D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S$.*
2. *Cone mapping: $-\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} S \subseteq K$.*

Then $\Delta y \equiv g(\lambda_1) - g(\lambda_0) \in K$.

Proof. Define $y(t) = g(\gamma(t))$. By the chain rule,

$$y'(t) = Dg(\gamma(t))\gamma'(t).$$

Because $D_y f(g(\gamma(t)), \gamma(t))$ is nonsingular for all $t \in [0, 1]$, the Jacobian of g along the path is

$$Dg(\lambda) = -\left[D_y f(g(\lambda), \lambda)\right]^{-1} D_\lambda f(g(\lambda), \lambda).$$

Hence,

$$y'(t) = -\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t).$$

By Assumption (1), $D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S$. Applying Assumption (2), it follows that $y'(t) \in K$ for all $t \in [0, 1]$. Therefore,

$$g(\lambda_1) - g(\lambda_0) = y(1) - y(0) = \int_0^1 y'(t)dt \in K$$

where the final conclusion follows from Lemma 5. □

Theorem 4 shows that global comparative statics reduces to verifying that local responses remain within a cone along an admissible path. It separates comparative statics into two components: the shock, captured by $D_\lambda f \gamma'(t)$, and the propagation mechanism, captured by $-[D_y f]^{-1}$. Any local comparative statics argument that can be expressed as

$$D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S \text{ and } -\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} S \subseteq K$$

extends immediately to finite parameter changes along any path where these conditions hold.

The contribution of Theorem 4 lies not in technical difficulty, but in identifying a general structure under which local comparative statics aggregate to finite changes. While the importance of "comparative statics in the large" has long been recognized (Samuelson, 1947), existing approaches typically rely on model-specific arguments or strong structural restrictions. Theorem 4 shows that, once a global equilibrium selection is available, finite-change comparative statics follow from a simple path integration argument under a cone invariance condition. This provides a unifying and broadly applicable framework that has been largely absent from the literature.

In multidimensional settings, global comparative statics is inherently path dependent, as there is generally no canonical path between two parameter values. If Λ is convex then a natural choice is the affine path $\gamma(t) = \lambda_0 + t(\lambda_1 - \lambda_0)$, in which case the shock condition becomes

$$D_\lambda f(g(\gamma(t)), \gamma(t))(\lambda_1 - \lambda_0) \in S.$$

However, the result requires only that the conditions hold along *some* admissible path. As a result, comparative statics conclusions depend on the existence of such paths rather than on any particular choice, and the affine path need not be admissible in every application.

In practice, admissibility is often ensured by imposing structural conditions that hold uniformly over a region of the parameter space, together with restrictions on the admissible directions of parameter change. The following corollary gives a simple case in which the affine path is admissible whenever the induced shock and cone-mapping conditions hold uniformly along each segment.

Corollary 1. *Suppose $\Lambda_0 \subseteq \Lambda$ is convex. Under the conditions of Theorem 4, suppose that for every ordered pair $\lambda_0, \lambda_1 \in \Lambda_0$, the affine path $\gamma(t) = \lambda_0 + t(\lambda_1 - \lambda_0)$ satisfies,* *for all* $t \in [0,1]$*,*

1. $D_\lambda f(g(\gamma(t)), \gamma(t))(\lambda_1 - \lambda_0) \in S$*,*

2. $-\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} S \subseteq K$*.*

Then, for every such ordered pair $\lambda_0, \lambda_1 \in \Lambda_0$*,* $\Delta y \equiv g(\lambda_1) - g(\lambda_0) \in K$*.*

Proof. Since Λ_0 is convex, the affine path lies in $\Lambda_0 \subseteq \Lambda$. Along this path, $\gamma'(t) = \lambda_1 - \lambda_0$. The stated assumptions are exactly the shock restriction and cone mapping condition in Theorem 4. The result follows immediately. □

5.1.1 Scalar parameter changes ($m = 1$)

Although economic models often involve many parameters, comparative statics exercises often isolate the change in a single scalar parameter. When $m = 1$, the notation simplifies and the result becomes a direct integration argument.

Corollary 2. *Let* $\Lambda \subseteq \mathbb{R}$*, and let* λ_0*,* $\lambda_1 \in \Lambda$ *with* $\lambda_1 \geq \lambda_0$ *and* $[\lambda_0, \lambda_1] \subseteq \Lambda$*. Under the conditions of Theorem 4, suppose that for all* $\lambda \in [\lambda_0, \lambda_1]$*,*

1. $D_\lambda f(g(\lambda), \lambda) \in S$*,*

2. $-[D_y f(g(\lambda), \lambda)]^{-1} S \subseteq K$*.*

Then, $\Delta y \equiv g(\lambda_1) - g(\lambda_0) \in K$*.*

Proof. Consider the affine path $\gamma(t) = \lambda_0 + t(\lambda_1 - \lambda_0)$. Since $[\lambda_0, \lambda_1] \subseteq \Lambda$, this path lies in Λ. Moreover, $\gamma'(t) = \lambda_1 - \lambda_0 \geq 0$. If $D_\lambda f(g(\lambda), \lambda) \in S$ for all $\lambda \in [\lambda_0, \lambda_1]$, then $D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S$ for all $t \in [0,1]$ because S is a cone. The cone mapping condition also holds along γ by assumption. Hence, the hypotheses of Theorem 4 hold along γ, and the conclusion follows. □

5.2 Necessary Conditions for Globalization

These results can also be reversed. A complementary necessity result shows that these conditions are not only sufficient but, under a local finite-change hypothesis, such global behavior implies corresponding pointwise restrictions on the Jacobian.

The necessity result identifies local restrictions on the Jacobian only along directions induced by admissible shocks; recovering a full cone-mapping property requires sufficiently rich variation in shocks.

The essential intuition is captured in the increasingness of a function. If $F(x)$ is continuously differentiable and $F'(x) \geq 0$ on $[x_0, x_1]$ then $F(x_1) \geq F(x_0)$, but the converse is not true in general. But if we define $x(t) = x_0 + t(x_1 - x_0)$ and assume that $F(x(t)) \geq F(x_0)$ for all sufficiently small $t > 0$, then differentiating with respect to t and evaluating at $t = 0$ gives $F'(x_0)(x_1 - x_0) \geq 0$. Thus, $F'(x_0) \geq 0$. To generalize this argument to multivariate implicit functions, the property must hold on all relevant paths between the two points.

Theorem 5. *Let $Y \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^m$ be open sets, and let $f : Y \times \Lambda \to \mathbb{R}^n$ be C^1. Suppose $g : \Lambda \to Y$ is C^1 and satisfies $f(g(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$, and that $D_y f(g(\lambda), \lambda)$ is nonsingular for every $\lambda \in \Lambda$. Let $G \subset \mathbb{R}^m$ and $S, K \subset \mathbb{R}^n$ be closed cones, and fix $\bar{\lambda} \in \Lambda$. Assume the following* **local finite change property** *holds at $\bar{\lambda}$:*

For every C^1 path $\gamma : [0, 1] \to \Lambda$ with $\gamma(0) = \bar{\lambda}$ and $\gamma'(t) \in G$ for all t, there exists $\varepsilon_\gamma \in (0, 1]$ such that, for every $\varepsilon \in (0, \varepsilon_\gamma]$,

$$D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S \ \forall t \in [0, \varepsilon] \implies g(\gamma(\varepsilon)) - g(\gamma(0)) \in K.$$

Then, for every $\eta \in G$,

$$D_\lambda f(g(\bar{\lambda}), \bar{\lambda})\eta \in \mathit{int}(S) \implies Dg(\bar{\lambda})\eta = -[D_y f(g(\bar{\lambda}), \bar{\lambda})]^{-1} D_\lambda f(g(\bar{\lambda}), \bar{\lambda})\eta \in K.$$

Proof. Fix $\eta \in G$ such that $D_\lambda f(g(\bar{\lambda}), \bar{\lambda})\eta \in \text{int}(S)$. We must show that $Dg(\bar{\lambda})\eta \in K$.

Consider the affine path $\gamma_r(t) = \bar{\lambda} + tr\eta$, $t \in [0, 1]$. Because Λ is open and $\bar{\lambda} \in \Lambda$, there exists $r > 0$ sufficiently small such that $\bar{\lambda} + tr\eta \in \Lambda$. Then $\gamma(0) = \bar{\lambda}$ and $\gamma'(t) = r\eta \in G$ because G is a cone.

Now define

$$\psi_r(t) \equiv D_\lambda f(g(\gamma_r(t)), \gamma_r(t))r\eta.$$

Since f and g are C^1, ψ_r is continuous. Moreover, $\psi_r(0) = rD_\lambda f(g(\bar{\lambda}), \bar{\lambda})\eta \in \text{int}(S)$ since S is a cone and $r > 0$. Hence, $\psi_r(t) \in S$ for sufficiently small $t \geq 0$.

By the local finite change property, for all sufficiently small $s > 0$,

$$g(\gamma_r(s)) - g(\bar{\lambda}) \in K.$$

Since K is a cone, dividing by $s > 0$ preserves membership:

$$\frac{g(\gamma_r(s)) - g(\bar{\lambda})}{s} \in K.$$

Because $g \circ \gamma_r$ is differentiable at 0 and K is closed, taking the limit as s goes to zero gives

$$\frac{d}{dt} g(\gamma_r(t))|_{t=0} \in K.$$

Using the chain rule,

$$\frac{d}{dt} g(\gamma_r(t))|_{t=0} = r Dg(\bar{\lambda})\eta.$$

Since K is a cone and $r > 0$, $Dg(\bar{\lambda})\eta \in K$. □

The requirement that $D_\lambda f(g(\bar{\lambda}), \bar{\lambda})\eta$ lie in the interior of S is essential. When shocks lie on the boundary of the cone, nearby perturbations may leave S, so local finite change behavior need only imply the cone mapping condition on the relevant face of S, rather than on the full cone.

5.3 Globalization Applications

Theorem 4 provides a unified way to globalize local comparative statics results. The following examples illustrate how a wide range of familiar comparative statics results can be recovered and extended within a single unified framework. In each application, the argument proceeds by identifying the shock cone S describing admissible perturbations of the system, and then verifying that the propagation operator $-[D_y f]^{-1}$ satisfies the cone mapping condition for an appropriate outcome change cone K. The flexibility in choosing K allows the framework to capture not only monotone responses but also relative dominance and inequality constraints across outcomes.

We illustrate the general necessity result from Theorem 5 in our first application, but subsequently focus on the sufficiency conditions of Theorem 4. In many cases, analogous necessity statements can be obtained by verifying the local finite change property.

5.3.1 Monotone Comparative Statics

A standard local sufficient condition for monotone comparative statics is entrywise nonnegativity of the inverse Jacobian, $-\left[D_y f(g(\lambda),\lambda)\right]^{-1} \geq 0$, which implies that nonnegative shocks weakly raise all equilibrium outcomes.

To apply Theorem 4, take the shock cone and outcome change cone to be the nonnegative orthant: $S = K = \mathbb{R}^n_+$. If parameter changes along a path γ generate shocks satisfying

$$D_\lambda f(g(\gamma(t)),\gamma(t))\gamma'(t) \in \mathbb{R}^n_+,$$

and the propagation operator satisfies

$$-\left[D_y f(g(\gamma(t)),\gamma(t))\right]^{-1} \mathbb{R}^n_+ \subseteq \mathbb{R}^n_+ \;\forall t \in [0,1],$$

then Theorem 4 implies

$$\Delta y \in \mathbb{R}^n_+,$$

so all outcomes weakly increase.

This condition is closely related to familiar complementarity restrictions. If $\partial f_i/\partial y_j \geq 0$ for all $i \neq j$, then $-D_y f$ is a Z-matrix. If, in addition, $-D_y f$ is nonsingular and $[-D_y f]^{-1} \geq 0$, then $-D_y f$ is an M-matrix.

When f is a system of first-order conditions from an optimization problem with objective function F, supermodularity corresponds to nonnegative cross-partials

$$\partial^2 F/\partial y_i \partial y_j = \partial f_i/\partial y_j \geq 0,$$

and increasing differences correspond to $\partial^2 F/\partial y_i \partial \lambda \equiv \partial f_i/\partial \lambda \geq 0$ for all i. Unlike lattice-based approaches, Theorem 4 applies on general open domains.

The monotone comparative statics result also admits a converse under the local finite change hypothesis of Theorem 5, provided shocks are sufficiently rich. Suppose that, for all parameter values $\lambda_1 \geq \lambda_0$, equilibrium outcomes satisfy $g(\lambda_1) \geq g(\lambda_0)$, and that this property holds locally along every sufficiently small admissible path segment. Let $G \subseteq \mathbb{R}^m_+$ denote the cone of admissible monotone parameter directions. The corresponding admissible shock cone generated by G is

$$D_\lambda f(g(\bar{\lambda}),\bar{\lambda})G.$$

If $D_\lambda f(g(\bar{\lambda}), \bar{\lambda}) > 0$, then every nonzero direction $\eta \in G$ generates an interior shock:

$$D_\lambda f(g(\bar{\lambda}), \bar{\lambda})\eta \in \mathbb{R}^n_{++} = \mathrm{int}(\mathbb{R}^n_+).$$

Theorem 5 therefore implies the local propagation restriction on the interior part of the admissible shock cone:

$$-[D_y f(g(\bar{\lambda}), \bar{\lambda})]^{-1} \left(D_\lambda f(g(\bar{\lambda}), \bar{\lambda}) G \cap \mathbb{R}^n_{++} \right) \subseteq \mathbb{R}^n_+.$$

If, in addition, the admissible shock cone is rich enough that

$$\mathbb{R}^n_{++} \subseteq D_\lambda f(g(\bar{\lambda}), \bar{\lambda}) G,$$

then

$$-[D_y f(g(\bar{\lambda}), \bar{\lambda})]^{-1} \mathbb{R}^n_{++} \subseteq \mathbb{R}^n_+.$$

By continuity this extends to the closure of the cone, yielding

$$-[D_y f(g(\bar{\lambda}), \bar{\lambda})]^{-1} \mathbb{R}^n_{+} \subseteq \mathbb{R}^n_+,$$

which is equivalent to entrywise nonnegativity of the inverse Jacobian. Thus, monotonicity of equilibrium outcomes with respect to sufficiently rich positive shocks implies that $[-D_y f]^{-1}$ is entrywise nonnegative. If, in addition, $-D_y f$ is known to be a Z-matrix, then it is also an M-matrix.

To see why the necessity direction requires strictly positive shocks, consider a two-dimensional system in which $D_\lambda f$ affects only the first equation, so shocks take the form $(s_1, 0)$. In this case, monotonicity of the equilibrium outcomes only requires that the first column of $-[D_y f]^{-1}$ be nonnegative; the second column plays no role. By contrast, if $D_\lambda f > 0$, shocks lie in the interior of $\mathbb{R}^2_+$, and monotonicity requires that $-[D_y f]^{-1}$ be entrywise nonnegative.

5.3.2 Shocks to a Single Equation

Many comparative statics questions concern the response of a particular outcome y_i to a shock that enters only equation i. Let

$$S_i = \{x \in \mathbb{R}^n : x_i \geq 0,\ x_j = 0 \text{ for } j \neq i\} \text{ and } K_i = \{x \in \mathbb{R}^n : x_i \geq 0\}.$$

If along a path γ,

$$D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S_i$$

and the cone mapping condition

$$-\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} S_i \subseteq K_i$$

holds for all t, then Theorem 4 implies $\Delta y \in K$, that is, y_i weakly increases with the finite parameter change.

5.3.3 Aggregate Outcomes and Causal Estimands

Many predictions concern aggregates rather than componentwise comparisons: totals, averages, or intergroup comparisons. The cone framework accommodates these cases by encoding restrictions on linear functionals of Δy.

Let $w \in \mathbb{R}^n$, and define the outcome change cone

$$K = \{x \in \mathbb{R}^n : w'x \geq 0\}.$$

Then $\Delta y \in K$ is equivalent to a nonnegative change in the linear functional $w'y$.

A particularly useful case arises in treatment effect settings. Let $a \in \{0,1\}^n$ denote treatment assignment, and define weights

$$w_i(a) = \begin{cases} \frac{1}{n_t} & \text{if } a_i = 1, \\ -\frac{1}{n_u} & \text{if } a_i = 0, \end{cases}$$

where $n_t = \sum_{i=1}^n a_i$ and $n_u = n - n_t$ denote the number of treated and untreated units, respectively. The cone

$$K(a) = \{x \in \mathbb{R}^n : w(a)'x \geq 0\} \tag{7}$$

encodes the condition that the average change among treated units weakly exceeds that among untreated units, which corresponds to the treatment effect studied in Christensen (2024).

Define the corresponding shock cone as

$$S(a) = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ if } a_i = 1, x_i = 0 \text{ if } a_i = 0\}.$$

If along a path γ, $D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S(a)$ and the cone mapping condition

$$-\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} S(a) \subseteq K(a)$$

holds for all t, then Theorem 4 implies $\Delta y \in K$. Thus, local conditions ensuring nonnegative treatment effects extend to finite parameter changes. This illustrates that the cone framework naturally accommodates causal estimands defined by linear functionals.

5.3.4 Linear Inequality Systems

More generally, any cone of the form

$$K = \{x \in \mathbb{R}^n : Wx \geq 0\}$$

for some matrix W is a closed convex cone. Such cones encode simultaneous sign restrictions on multiple linear combinations of outcomes. Such cones allow comparative static statements such as: "overall average increases", "average outcomes among the treated increase," "dispersion decreases", or "a collection of subgroup means rises." Theorem 4 then globalizes these restrictions to finite parameter changes whenever the corresponding cone mapping condition $-[D_y f]^{-1} S \subseteq K$ holds along a path.

5.3.5 Bounds and Relative Comparisons

Some comparative statics results provide quantitative bounds or relative comparisons across equilibrium outcomes (e.g., Christensen, 2024; Norris, Johnson, and Spitkovsky, 2023). These can be incorporated into the cone framework by defining cones that encode the relevant inequalities.

Relative Comparisons Across Outcomes. Suppose local conditions imply

$$g_i'(\lambda) \geq \left|g_j'(\lambda)\right|.$$

Define the cone

$$K_{ij} = \{x \in \mathbb{R}^n : x_i \geq |x_j|\}.$$

If along a path γ, $D_\lambda f(g(\gamma(t)), \gamma(t))\gamma'(t) \in S$ and the cone mapping condition

$$-\left[D_y f(g(\gamma(t)), \gamma(t))\right]^{-1} S \subseteq K_{ij}$$

holds for all t, then Theorem 4 implies $\Delta y \in K_{ij}$, so that $g_i(\lambda_1) - g_i(\lambda_0) \geq |g_j(\lambda_1) - g_j(\lambda_0)|$.

Absolute Bounds. Suppose that along a path γ, local conditions imply a lower bound on the response of outcome i. In the scalar parameter case, this takes the form:

$$g_i'(\lambda) \geq c > 0.$$

The cone framework extends immediately to such affine restrictions.

While the natural outcome set $\{x \in \mathbb{R}^n : x_i \geq c\}$ is not a cone, Lemma 5 can still be used after subtracting the affine term. Let $h(t) = y'(t) - b(t)$ and suppose local conditions imply $h(t) \in K$ along an admissible path. Then integration gives

$$g(\lambda_1) - g(\lambda_0) - \int_0^1 b(t)dt \in K.$$

Thus, lower bounds on marginal responses globalize to lower bounds on finite changes.

Specifically, in the scalar parameter case ($m = 1$) with $\lambda_1 \geq \lambda_0$, take $K_i = \{x \in \mathbb{R}^n : x_i \geq 0\}$ and $b(t) = c\gamma'(t)e_i$. If

$$g_i'(\gamma(t))\gamma'(t) \geq c\gamma'(t)$$

along the path, then

$$g(\lambda_1) - g(\lambda_0) - c(\lambda_1 - \lambda_0)e_i \in K_i.$$

Thus, $g_i(\lambda_1) - g_i(\lambda_0) \geq c(\lambda_1 - \lambda_0)$.

5.4 Discussion

This section shows that global comparative statics can be obtained by combining global equilibrium selection with cone-based restrictions on local responses. The implicit function theorem yields a linear mapping from parameter changes to equilibrium outcomes, and cones provide a flexible way to encode economically meaningful restrictions on both shocks and responses. When these restrictions are preserved along a path in parameter space, local comparative statics results extend directly to finite changes. This approach unifies a wide range of comparative statics statements—including monotonicity, targeted shocks, and inequality constraints—without requiring lattice structure or rectangular domains.

The necessity result in Theorem 5 complements this analysis by showing that, under a local finite change hypothesis, global comparative statics behavior also restricts the local structure of the system. In particular, it implies that if equilibrium responses satisfy a given inequality for all sufficiently small admissible parameter changes, then the corresponding cone mapping condition must hold pointwise. Thus, the framework not only globalizes local comparative statics, but also clarifies the extent to which global behavior reflects underlying Jacobian structure.

6 Application: Differentiated Products Pricing

We now illustrate our results in a differentiated products pricing model, a canonical setting in empirical industrial organization. Firms set prices to maximize profits, leading to a system of first-order conditions that implicitly defines equilibrium prices as a function of costs and demand parameters. The goal is not to develop new local comparative statics results for pricing games, but to show how familiar local restrictions can be combined with the results above to obtain finite-change predictions. This setting therefore provides a useful benchmark for the operational content of the global comparative statics results.

6.1 Setup

Consider a market with J products indexed by $j = 1, ..., J$. Let $p \in \mathcal{Y} \subseteq \mathbb{R}^J$ denote prices, and let $\theta \in \Lambda \subseteq \mathbb{R}^m$ denote demand or cost shifters. Let $s_j(p, \theta)$ denote demand (or market share) for product j, and let $c_j(\theta)$ denote marginal cost. Firms choose prices to maximize profits, leading to the system of first-order conditions

$$f_j(p, \theta) = s_j(p, \theta) + \sum_{k=1}^{J} \Omega_{jk}(p_k - c_k(\theta)) \frac{\partial s_k(p, \theta)}{\partial p_j} = 0, \;\; j = 1, ..., J,$$

where Ω is the ownership matrix, with $\Omega_{jk} = 1$ if products j and k are owned by the same firm and $\Omega_{jk} = 0$ otherwise. The single-product firm case corresponds to $\Omega = I$. Let $f(p, \theta) = (f_1, ..., f_J)'$. Equilibrium prices are implicitly defined by

$$f(p, \theta) = 0.$$

6.2 Global Equilibrium

We first discuss conditions under which equilibrium prices are uniquely determined as a smooth function of parameters. The Jacobian matrix $D_p f(p, \theta)$ captures how marginal profitability responds to price changes. Its structure reflects substitution patterns across products. In particular, own-price effects determine diagonal elements, while cross effects determine off-diagonal entries. In many differentiated product models, demand satisfies:

1. $\frac{\partial s_j}{\partial p_j} < 0$,
2. $\frac{\partial s_i}{\partial p_j} \geq 0$ for $i \neq j$.

Under additional curvature restrictions ensuring that $\partial f_j / \partial p_i \geq 0$ for $i \neq j$ and $\partial f_j / \partial p_j < 0$, the matrix $-D_p f(p, \theta)$ is a Z-matrix with positive diagonal.

We now provide a primitive sufficient condition for nonsingularity and a uniform bound on the inverse Jacobian.

Proposition 3. *Suppose that for all (p, θ),*

1. *$-D_p f(p, \theta)$ is a Z-matrix, and*
2. *there exists $\varepsilon > 0$ such that $-D_p f(p, \theta)\mathbf{1} \geq \varepsilon \mathbf{1}$.*

Then $-D_p f(p, \theta)$ is a nonsingular M-matrix for all (p, θ) and $\|[D_p f(p, \theta)]^{-1}\|_\infty \leq \frac{1}{\varepsilon}$.

Proof. Assumptions 1-2 imply that $-D_p f$ is a Z-matrix with strictly positive row sums. It follows that $-D_p f(p, \theta)$ is a nonsingular M-matrix (see Plemmons, 1977). The bound on the inverse follows from the uniform lower bound on row sums, which implies that the infinity norm of the inverse is bounded by $1/\varepsilon$, as shown in the discussion following Proposition 2. Specifically, since $A = -D_p f$ is a nonsingular M-matrix, $A^{-1} \geq 0$. Moreover $A\mathbf{1} \geq \varepsilon \mathbf{1}$ implies $A^{-1} A \mathbf{1} \geq \varepsilon A^{-1} \mathbf{1}$, hence $\mathbf{1} \geq \varepsilon A^{-1} \mathbf{1}$. Therefore, $\|A^{-1}\|_\infty \leq 1/\varepsilon$. Since $[D_p f]^{-1} = -A^{-1}$, the same bound holds. $\square$

The row sum condition in this result requires that under a uniform increase in all prices, the own-effect on marginal profit dominates cross price effects by a uniform margin for all firms.

Proposition 2 provides a convenient route toward establishing the existence of a global equilibrium mapping from parameter space into prices. In particular, if $\mathcal{Y} = \mathbb{R}^J$, $\Lambda = \mathbb{R}^m$, and (1) $-D_p f(p, \theta)$ is everywhere nonsingular, (2) $\|[D_p f(p, \theta)]^{-1}\|$ is uniformly bounded, and (3) $\|D_\theta f(p, \theta)\|$ is uniformly bounded, then the augmented mapping

$$\phi(p, \theta) = (-f(p, \theta), \theta)$$

is a diffeomorphism, and there exists a unique continuously differentiable equilibrium mapping $p = g(\theta)$.

Proposition 3 provides primitive conditions ensuring (1)-(2). The boundedness condition on $D_\theta f$ is mild for many economically relevant parameters. For example, if θ is a cost shifter entering marginal costs additively and demand derivatives are bounded, then $D_\theta f$ is uniformly bounded in standard differentiated products demand systems.

Since Proposition 3 is stated on $\mathbb{R}^J$, an additional condition is needed to ensure that equilibrium prices are economically meaningful. We impose the following condition: *if $p \notin \mathbb{R}^J_{++}$, then there is some firm k such that $f_k(p, \theta) > 0$.*[3] This condition rules out any equilibrium with a zero or negative price, since at such a profile at least one firm has a profitable upward price deviation. Consequently, any solution to $f(p, \theta) = 0$ must satisfy $p \in \mathbb{R}^J_{++}$.

Under these conditions, it follows from Proposition 2 that there exists a continuously differentiable equilibrium price mapping $g : \mathbb{R}^m \to \mathbb{R}^J_{++}$ satisfying $f(g(\theta), \theta) = 0$ for all $\theta \in \mathbb{R}^m$.

An alternative route to this conclusion runs through the global injectivity results from Section 4. For each fixed θ, define the slice map $F_\theta(p) = -f(p, \theta)$. Since M-matrices are P-matrices, the assumptions of Proposition 3 imply that $D_p F_\theta(p) = -D_p f(p, \theta)$ satisfies the weak P-matrix condition on the open rectangular domain $\mathbb{R}^J$. Thus, Theorem 2 establishes uniqueness of equilibrium prices, provided existence is assumed or established separately. The connected substitutes route in Theorem 3 would require additional structure, such as weak column diagonal dominance

[3] More generally, there is some firm k such that $f_k(p, \theta) \neq 0$.

of $-D_p f(p, \theta)$. These injectivity arguments establish uniqueness conditional on existence. By contrast, Proposition 2 provides existence and uniqueness directly under the uniform inverse-bound and parameter-derivative conditions.

6.3 Global Comparative Statics

We now apply Theorem 4 to characterize how equilibrium prices respond to parameter changes. Let $\theta_0, \theta_1 \in \mathbb{R}^m$, and let $\gamma : [0,1] \to \mathbb{R}^m$ be a C^1 path connecting them.

Define $p(t) = g(\gamma(t))$. By the IFT,

$$Dg(\theta) = -[D_p f(p, \theta)]^{-1} D_\theta f(p, \theta).$$

Local comparative statics results for pricing games are well understood. Under the conditions in Proposition 3, or other standard assumptions on curvature and substitution patterns (e.g., see Vives, 1999), the inverse Jacobian is nonnegative:

$$-[D_p f(p, \theta)]^{-1} \geq 0.$$

Thus, cost changes that cause $D_\theta f(p, \theta) \geq 0$ lead to nonnegative price responses. Our contribution is to show that such local results extend to finite parameter changes, provided the corresponding cone mapping condition holds along an admissible path. For simplicity, consider the straight line path $\gamma(t) = \theta_0 + t(\theta_1 - \theta_0)$ and suppose $\theta_1 \geq \theta_0$ represents a cost increase.

6.3.1 Qualitative Impacts

To sign the impact of a cost increase, note that $D_\theta f(p, \theta) \geq 0$ implies

$$D_\theta f(p(t), \gamma(t))(\theta_1 - \theta_0) \in \mathbb{R}^J_+,$$

so the shock cone is $S = R^J_+$.

If $-D_p f(p(t), \gamma(t))$ is an M-matrix for all t, we have

$$-[D_p f(p(t), \gamma(t))]^{-1} \mathbb{R}^J_+ \subseteq \mathbb{R}^J_+ \ \ \forall t \in [0, 1].$$

Then Theorem 4 implies $g(\theta_1) \geq g(\theta_0)$. Thus, prices weakly increase in response to finite cost increases.

6.3.2 Relative Price Responses

The framework also yields predictions about relative price changes across products. Suppose that one component of θ is a cost shifter c_i, or that the path γ induces an increase only in firm i's marginal cost. Suppose local conditions imply

$$\frac{\partial p_i}{\partial c_i} \geq \left| \frac{\partial p_j}{\partial c_i} \right| \; \forall j \neq i.$$

Define the cone $K_{ij} = \{x \in \mathbb{R}^J : x_i \geq |x_j|\}$. If the induced shock lies in S_i and the cone mapping condition

$$-[D_p f(p(t), \gamma(t))]^{-1} S_i \subseteq K_{ij}$$

holds along the path, Theorem 4 implies $g_i(\theta_1) - g_i(\theta_0) \geq |g_j(\theta_1) - g_j(\theta_0)|$ for $j \neq i$. By Theorem 4 in Christensen (2024), if $-D_p f(p(t), \gamma(t))$ is a Z-matrix and a B-matrix for all t, then the cone mapping condition is satisfied.[4]

6.3.3 Cost Pass-through

An important question in industrial organization concerns the magnitude of pass-through—how much prices increase in response to cost increases. In the scalar parameter case, suppose that local conditions, such as those in Norris, Johnson, and Spitkovsky, 2023, imply $Dg_i(\theta) \geq c > 0$ along the path. Then Corollary 2 yields

$$g_i(\theta_1) - g_i(\theta_0) \geq c(\theta_1 - \theta_0).$$

More generally, the bounding arguments in Section 5.3.5 extend local pass-through bounds to finite changes whenever the relevant inequalities hold along the admissible path.

[4]The same conditions also imply inverse positivity of $-D_p f$. Together with the global selection conditions in Section 6.2, this yields a smooth equilibrium function.

6.3.4 Average Treatment Effects

Reduced form empirical techniques typically test for average effects of shocks. If a cost shock applies to a subset T of products, the cone K can encode aggregate comparisons such as

$$\frac{1}{|T|}\sum_{j\in T}\Delta p_j \geq \frac{1}{|U|}\sum_{j\in U}\Delta p_j,$$

where $U = J \setminus T$. This inequality says that the average price change among directly affected products is greater than the average change among products not directly affected by the cost shock. The relevant outcome change cone is defined in (7) and the aggregate cone results in Section 5.3.3 translate local predictions about relative pass-through into finite change predictions. See Christensen (2024) for a detailed discussion of this case.

6.4 Discussion

This example illustrates how the framework developed in Sections 3-5 applies in a canonical economic setting. The pricing equilibrium is defined by a system of nonlinear equations with rich cross-effects, and global equilibrium existence and uniqueness follow from matrix conditions on $D_p f$. Global comparative statics are obtained by verifying that local responses satisfy a cone mapping condition along an admissible path. While the example focused on comparative statics of descriptive outcomes, the same methods apply equally well to normative questions.

The preceding examples illustrate why the cone formulation is useful even when the underlying outcome space admits a componentwise order. Many comparative statics predictions in pricing models are not pure monotonicity statements. They compare magnitudes, averages, or weighted responses across products. Lattice-based methods are designed primarily to establish monotone movement with respect to an order, whereas the present framework allows the relevant order to be chosen directly from the economic prediction. The same smooth equilibrium system can therefore generate componentwise, relative, and aggregate finite-change predictions depending on the cone used to encode the conclusion.

7 Conclusion

This paper develops a framework for extending local comparative statics results based on the implicit function theorem to finite parameter changes. By combining global inversion results with a path-integration argument, we show that local derivative information can be used to derive global predictions under a cone invariance condition. The equilibrium selection results provide conditions under which the global mapping exists, while the cone integration results characterize how that mapping changes over finite parameter movements.

The results clarify when local comparative statics are sufficient for understanding the effects of finite changes in parameters, as commonly encountered in applied and quantitative work. In addition, the necessity result shows that, for interior shocks, global monotonicity imposes sharp restrictions on the Jacobian of the system.

The framework complements existing approaches based on lattice structure by applying to smooth models on general domains and by directly connecting local derivatives to finite changes. Future work may extend these ideas to settings with non-smooth equilibria or correspondence valued solutions.

A Appendix

Proof of Lemma 3. ($\Rightarrow$)Suppose ϕ is proper. Fix a compact set $C \subseteq \Lambda$. Let $K \subseteq \mathbb{R}^n$ be compact. Define the set $U \equiv (-K) \times C \subseteq \mathbb{R}^n \times \Lambda$, which is compact. Then

$$\begin{aligned}\phi^{-1}(U) &= \{(y, \lambda) \in \mathcal{Y} \times \Lambda : -f(y, \lambda) \in -K, \lambda \in C\} \\ &= \{(y, \lambda) \in \mathcal{Y} \times C : f(y, \lambda) \in K\} \\ &= f_C^{-1}(K).\end{aligned}$$

Since ϕ is proper and U is compact, $\phi^{-1}(U)$ is compact. Hence $f_C^{-1}(K)$ is compact. Because K was arbitrary, f_C is proper.

($\Leftarrow$) Suppose f_C is proper for every compact set $C \subseteq \Lambda$. Let $U \subseteq \mathbb{R}^n \times \Lambda$ be compact. Set $C = \pi_\Lambda(U)$ and $K = \pi_{\mathbb{R}^n}(U)$. Both C and K are compact because projections are continuous. Since $U \subseteq K \times C$, $\phi^{-1}(U) \subseteq \phi^{-1}(K \times C)$. Moreover,

$$\begin{aligned}\phi^{-1}(K \times C) &= \{(y, \lambda) \in \mathcal{Y} \times \Lambda : -f(y, \lambda) \in K, \lambda \in C\} \\ &= \{(y, \lambda) \in \mathcal{Y} \times C : f(y, \lambda) \in -K\} \\ &= f_C^{-1}(-K).\end{aligned}$$

Since $-K$ is compact and f_C is proper, $f_C^{-1}(-K)$ is compact.

Finally, $\phi^{-1}(U)$ is closed because U is closed and ϕ is continuous. Therefore, $\phi^{-1}(U)$ is a closed subset of the compact set $f_C^{-1}(-K)$, and hence compact. Therefore, ϕ is proper. $\square$

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