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A Necessary and Sufficient Condition for a Unique Maximum with an Application to Potential Games

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Abstract

Under regularity and boundary conditions which ensure an interior maximum, I show that there is a unique critical point which is a global maximum if and only if the Hessian determinant of the negated objective function is strictly positive at any critical point. Within the large class of Morse functions, and subject to boundary conditions, this *local* and *ordinal* condition generalizes strict concavity, and is satisfied by nearly all strictly quasiconcave functions. The result also provides a new uniqueness theorem for potential games.

Keywords: optimization, index theory, potential games

JEL codes: C02, C72

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1 Introduction

In many applied theory models, the analyst is interested in minimal conditions under which an optimization problem has a unique interior maximum. Once existence and interiority are established, the standard assumption which guarantees uniqueness is that the objective function is strictly quasiconcave, but this is not necessary. Under conditions which guarantee existence and interiority, and a mild regularity condition, I show that a necessary and sufficient condition for a function to have a unique critical point which is also a maximum is that the Hessian determinant of the negated objective function is strictly positive at any critical point. One especially notable aspect of this condition is that it is a *local* condition, unlike strict quasiconcavity which is a global property of the function.

The fact that the maximizer is also the unique critical point makes it relevant for potential games (see Monderer and Shapley, 1996). It is well-known that games with a strictly concave potential function have a unique equilibrium since these functions have at most one critical point which must be a maximum (see, for example, Neyman, 1997). Theorem 2 provides conditions under which strict concavity may be significantly weakened while still guaranteeing a unique Nash equilibrium.

The analysis relies on index theory, and the Poincaré-Hopf theorem in particular (e.g., Milnor, 1965; p. 35). Index theory has been applied fruitfully in general equilibrium theory, game theory, and equilibrium systems more generally.[1] In this note I demonstrate its usefulness in unconstrained optimization.

2 The Main Result

The main concern in this note is to prove and explore a few implications of the following Theorem, especially the second part. The result is proved in the next section.

Theorem 1 *Let $f : A \to \mathbb{R}$ be a Morse function defined on a contractible and compact smooth manifold $A \subset \mathbb{R}^n$. If A has boundary, then assume that ∇f is well-*

[1] A far from exhaustive list of references includes Dierker (1972), Varian (1974), and Kehoe (1985), for general equilibrium, Kolstad and Mathiesen (1987) and Hefti (2016) for game theory, and Dohtani (1998) and Christensen and Cornwell (2017) for equilibrium systems more generally.

defined[2] *and inward pointing on the boundary.*

1. *The number of critical points in A is finite and odd, and at least one of them is a maximum.*

2. *f has a unique critical point in A which is a global maximum if and only if*

$$\nabla f(x^*) = 0 \quad \Rightarrow \; \det\left(-D^2 f(x^*)\right) > 0. \tag{1}$$

Before addressing some of the less familiar assumptions of the Theorem, I will review some well-known optimization concepts to help put criterion (1) in context (e.g., Simon and Blume, 1994). A *critical point* x^* of a smooth function f is a point where the gradient vanishes, $\nabla f(x^*) = 0$. A *nondegenerate critical point* is a critical point where the Hessian determinant is nonsingular, $\det(D^2 f(x^*)) \neq 0$.[3] Any interior maximum must be a critical point, and the Hessian at an interior maximum is negative semidefinite, which implies $\det(-D^2 f(x^*)) \geq 0$. If f is globally strictly concave, then a critical point x^* is a global maximum. A sufficient condition for global concavity is that the Hessian of f is everywhere negative definite, and this requires $\det(-D^2 f(x)) > 0$ for all $x \in A$. The function $f(x) = -x^4$ provides a convenient reminder that negative definiteness is not necessary for a maximum or strict concavity.

However, in the class of *Morse functions*—smooth real-valued functions on a manifold A whose critical points are all nondegenerate—the condition $\det(-D^2 f(x^*)) > 0$ *is* necessary.[4] The class of Morse functions is large—it is well-known that Morse functions are generic in that they form an open, dense subset of all smooth functions $A \to \mathbb{R}$. In this sense, confining attention to the class of Morse functions is not very restrictive.

Clearly, the class of functions which satisfy criterion (1) contains the class of strictly concave Morse functions. Verifying criterion (1) is also simpler in the sense that it requires only information about the Hessian determinant and not the other

[2] A simple way to make this precise is to assume f is defined on an open set $X \subset \mathbb{R}^n$, where $A \subset X$.

[3] Alternative and slightly more appropriate notation for the Hessian in this setting would be $D_x \nabla f(x)$, which emphasizes that the Hessian is the Jacobian of $\nabla f(x)$. However, $D^2 f(x)$ is the more common notation, so I follow that convention here.

[4] Note that for multivariate functions this does not rule out that the Hessian may be negative semidefinite at x^*.

leading principal minors. Perhaps more significantly, in contrast to strictly concave Morse functions, condition (1) requires that $\det(-D^2 f(x)) > 0$ only at any critical point x^* rather than for all $x \in A$, so this is a *local* condition which, in combination with information on how the function behaves at its boundary, is sufficient to draw conclusions about the function's *global* characteristics.

The other assumptions of Theorem 1 are that A is a contractible, compact, and smooth manifold; and that the gradient points inward on the boundary of A. Since every convex set in $\mathbb{R}^n$ is contractible, the assumption that A is contractible, compact, and smooth manifold could be replaced with the standard assumption that $A \subset \mathbb{R}^n$ is convex and compact as long as the boundary of A is piecewise smooth. Loosely speaking, a contractible set is one that can be continuously shrunk to a point. In one dimension, only the closed interval is a contractible and compact manifold.[5] In higher dimensions, manifolds that are diffeomorphic to the unit disk $\{x \in \mathbb{R}^n | \sum x_i^2 \leq 1\}$ are contractible, compact, and smooth. By a smoothing argument along the lines of Section 3.1 in Christensen and Cornwell (2017), Theorem 1 also applies to manifolds with "corners" which often arise in applications, such as the solid rectangle, the simplex, or any other convex set with a piecewise smooth boundary.

As for the assumption that the gradient points inward on the boundary, recall that the gradient points in the direction of steepest ascent. Thus, this assumption means that from any point on the boundary, there is a way to move to the interior such that the function's value increases. Hence, a maximum cannot exist on the boundary of A. Formally, say that ∇f is *inward pointing on* ∂A if for any $\bar{x} \in \partial A$, there is some $\varepsilon_0 > 0$ such that $\bar{x} + \varepsilon \nabla f(\bar{x}) \in int(A)$ for all $0 < \varepsilon < \varepsilon_0$. We say that ∇f is *outward pointing* if $-\nabla f$ is inward pointing.

The boundary condition is often satisfied under standard conditions which ensure an interior maximum, as I now demonstrate in the following example.

Example 1 Consider the problem

$$\max_{x \in X} f(x),$$

where $X \subset \mathbb{R}^n$ and f is a Morse function. Suppose there is a (solid) rectangle

$$A = \{x \in X | a_i \leq x_i \leq b_i \text{ with } a_i, b_i \in \mathbb{R},\ i = 1, ..., n\}$$

[5] The circle is a compact manifold without boundary, but it is not contractible.

such that for any $x \in A$, and $i = 1, ..., n$,

$$\frac{\partial f(x)}{\partial x_i} > 0 \text{ if } x_i = a_i \text{ and} \tag{2}$$

$$\frac{\partial f(x)}{\partial x_i} < 0 \text{ if } x_i = b_i. \tag{3}$$

If $X = \mathbb{R}^n_+$, as is typical in industrial organization settings, then we usually have $a_i = 0$, and the existence of an upper bound b_i implies that the optimal value of any choice variable in A is not infinite.

The rectangle A is a compact and contractible manifold with boundary, and conditions (2) and (3) ensure that the boundary condition is satisfied.[6] The smoothing argument in Section 3.1 of Christensen and Cornwell (2017) implies that we can "round off" the corners of A and treat it as a smooth manifold. Theorem 1 then states that, on the domain A, there is a unique maximizer $x^* \in int(A)$ which is also the unique critical point if and only if $\nabla f(x^*) = 0$ implies $\det(-D^2 f(x^*)) > 0$.[7] To guarantee a strictly positive determinant, one can impose "strict mean positive dominance" on the Hessian of the negated objective function, $-f$, an economically meaningful restriction which also comes with nice comparative statics properties (Christensen, 2017). ■

In the preceding example, a standard and alternative assumption which guarantees uniqueness is that f is strictly quasiconcave. I now explore the relationship between strict quasiconcavity and functions which satisfy the conditions of Theorem 1. Let $\mathcal{E} = \{x^* \in A | \nabla f(x^*) = 0\}$ be the set of critical points of the function $f : A \to \mathbb{R}$.

Lemma 1 *Let $f : A \to \mathbb{R}$ be a Morse function on a smooth, compact manifold $A \subset \mathbb{R}^n$. Then $\mathcal{E}$ is finite.*

[6] Another natural way to define A in such a way that meets the boundary condition is to use an upper contour set,

$$A = \{x \in X | f(x) \geq c \text{ for } c \in \mathbb{R}\},$$

provided that this upper contour set is contractible and the level curve $\{x \in X | f(x) = c\}$ does not define a "plateau" where $\nabla f(x) = 0$ is possible.

[7] This guarantees a unique maximizer on A, not necessarily on X. To ensure a unique maximum on X it would be sufficient in this case to assume, for all i, $\frac{\partial f(x)}{\partial x_i} > 0$ if $x_i \leq a_i$ and $\frac{\partial f(x)}{\partial x_i} < 0$ if $x_i \geq b_i$.

Proof. Since $\det\left(-D^2 f\left(x^*\right)\right) \neq 0$, the Inverse Function Theorem implies ∇f is one-to-one in a neighborhood of each $x^* \in A$. Hence, the critical points are isolated.[8] Note that A is compact, so every infinite subset of A must contain at least one point of accumulation in A (e.g., Corollary 5.9 in Mendelson, 1990). It follows that $\mathcal{E}$ is finite since $\mathcal{E} \subset A$ contains only isolated points. ■

Proposition 1 *Let $f : A \rightarrow \mathbb{R}$ be defined on a convex and compact set $A \subset \mathbb{R}^n$, and assume that $\nabla f(x)$ is well-defined and inward pointing on the boundary of A. If f is a strictly quasiconcave Morse function then f has a unique global maximizer $x^* \in int(A)$ where $\nabla f\left(x^*\right) = 0$ and $\det\left(-D^2 f\left(x^*\right)\right) > 0$. Moreover, the number of critical points is finite, and, except for the global maximum, no critical point is an extremum.*

Proof. The existence of a unique global and interior maximum x^* where $\nabla f\left(x^*\right) = 0$ and $\det\left(-D^2 f\left(x^*\right)\right) \geq 0$ comes from the boundary condition, Weierstrauss' Theorem, and strict quasiconcavity. Then $\det\left(-D^2 f\left(x^*\right)\right) > 0$ follows by the definition of Morse functions. Finiteness follows from Lemma 1.

Let x_1 be an arbitrary critical point. Note that $x_1 \in int(A)$ by the boundary condition. x_1 cannot be a minimum, because if it were, then there would be an open ball $B\left(x_1, r\right) \subset A$ around x_1 with radius $r > 0$ such that $f\left(x_1\right) \leq f(y)$ for any $y \in B\left(x_1, r\right)$. Since the ball is convex, there would be some $y, y' \in B\left(x_1, r\right)$ and $\lambda \in (0,1)$ such that $x_1 = \lambda y + (1-\lambda) y'$. But strict quasiconcavity requires

$$f\left(x_1\right) = f\left(\lambda y + (1-\lambda) y'\right) > \min\left\{f(y), f\left(y'\right)\right\},$$

which would contradict x_1 being a minimum.

If $x_1 \neq x^*$, then x_1 cannot be a maximum. Since $f\left(x^*\right) > f\left(x_1\right)$, strict quasiconcavity implies

$$f\left(\lambda x_1 + (1-\lambda) x^*\right) > f\left(x_1\right) \text{ for any } \lambda \in (0,1).$$

By picking λ arbitrarily close to one, we can always find a point $\lambda x_1 + (1-\lambda) x^*$ arbitrarily close to x_1 where the function's value is higher than it is at x_1. ■

Proposition 1 demonstrates that if the boundary condition is satisfied, then the class of functions which satisfy criterion (1) "nearly" contains the class of strictly

[8] This is also a well-known consequence of the Morse lemma.

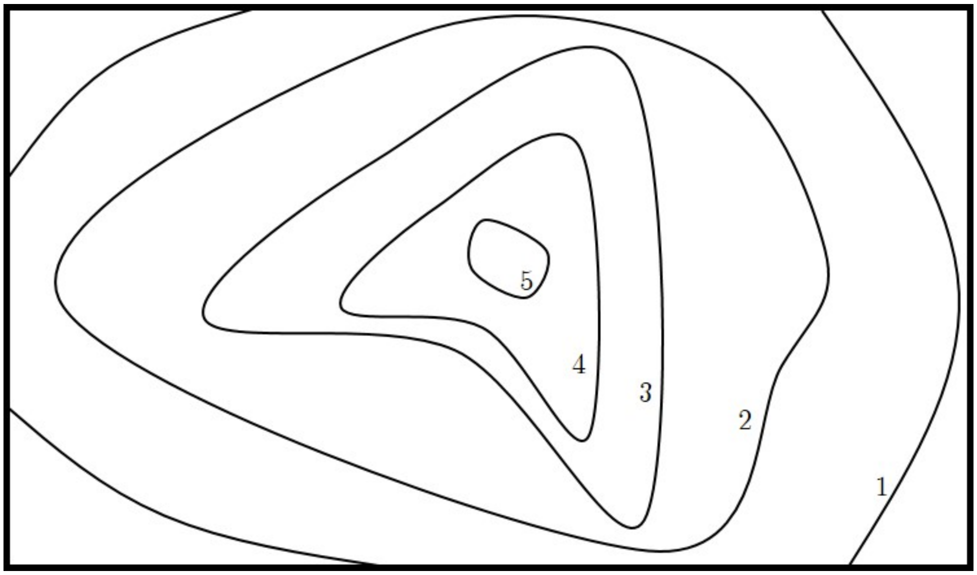

Figure 1: This function is not quasiconcave but is consistent with criterion (1).

quasiconcave Morse functions. The reason for the qualifier "nearly" is that the critical points of the latter may not be unique. However, these critical points are not important in the sense that there can only be a finite number of them and they cannot be local extrema.

In the other direction, the class of functions which satisfy criterion (1) is larger than the class of strictly quasiconcave Morse functions with a unique critical point. This is illustrated in Figure 1 which depicts the level curves of a function. The upper contour sets are not convex so the function is not strictly quasiconcave, yet the function is consistent with criterion (1) and the boundary condition.

Fortunately, however, criterion (1) retains the ordinal quality of quasiconcavity. The proof of this point also illustrates how the local nature of criterion (1) plays an important role.

Proposition 2 *Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth, strictly increasing function. Then $f(x) = h(g(x))$ meets criterion* (1) *iff g meets criterion* (1).

Proof. If $\frac{\partial f(x^*)}{\partial x_i} = \frac{\partial h}{\partial g}\frac{\partial g(x^*)}{\partial x_i} = 0$, the typical (i,j) element of the Hessian of f at a x^* is $\frac{\partial^2 f(x^*)}{\partial x_i \partial x_j} = \frac{\partial h}{\partial g}\frac{\partial^2 g}{\partial x_i \partial x_j}$. Then $D^2 f(x^*) = \frac{\partial h}{\partial g} D^2 g(x^*)$, so that $\det(-D^2 f(x^*)) = \left(\frac{\partial h}{\partial g}\right)^n \det(-D^2 g(x^*))$. ■

3 Proof of Theorem 1

The proof of part 1 borrows heavily from the proof of part 1 of Theorem 2 in Christensen and Cornwell (2017).

(1) Finiteness follows from Lemma 1.

The gradient $-\nabla f$ defines a smooth vector field over A which is outward pointing on the boundary of A. Moreover, the zeros of $-\nabla f$ (i.e., the critical points of f) are isolated by the same argument as in the proof of Lemma 1. Hence, the Poincaré-Hopf Theorem states that the index sum is equal to the Euler characteristic of A, which is $+1$ since A is a contractible subset of $\mathbb{R}^n$. The index of a critical point is $+1$ if $\det\left(-D^2 f\left(x^*\right)\right) > 0$ and -1 if $\det\left(-D^2 f\left(x^*\right)\right) < 0$. Since $\det\left(-D^2 f\left(x^*\right)\right) \neq 0$ and the index sum is $+1$, there must be an odd number of critical points.

Finally, by Weierstrauss' Theorem, f attains a maximum. The maximum must lie in the interior of A since ∇f is inward pointing on the boundary. Hence, at least one critical point is a maximum.

(2) Confine attention to interior maxima since maxima cannot lie on the boundary.

($\Rightarrow$) Any interior maximum x^* must be a critical point and $D^2 f\left(x^*\right)$ must be negative semidefinite. In general this implies $\det\left(-D^2 f\left(x^*\right)\right) \geq 0$, but for Morse functions we have $\det\left(-D^2 f\left(x^*\right)\right) \neq 0$.

($\Leftarrow$) Suppose $\det\left(-D^2 f\left(x^*\right)\right) > 0$ at each critical point x^* of f. The index at each of these critical points is $+1$, and since the index sum is $+1$, there can only be one. This completes the proof of Theorem 1.

It is now immediate that if in Theorem 1 we everywhere replace $\det\left(-D^2 f\left(x^*\right)\right)$ with $\det(D^2 f\left(x^*\right))$ and $-\nabla f$ with ∇f, then we can replace the word "maximum" with "minimum."

4 Application to Potential Games

Consider a game $\Gamma = \left(N; \left(X_i\right)_{i\in N}; \left(u^i\right)_{i\in N}\right)$ where N is the set of players, $X_i \subset \mathbb{R}^{m_i}$ is the strategy set for player i, and $u^i : X \to \mathbb{R}$ is player $i's$ payoff function where $X = \times_{i\in N} X^i$. The strategy profile x^* is a *pure strategy Nash equilibrium* if, for every player $i \in N$, $x_i^* \in \arg\max\left\{u^i\left(x_i, x_{-i}^*\right) | x_i \in X_i\right\}$.[9]

[9] As usual, the notation x_{-i} is a vector containing the strategies of every player but player i.

A *potential function* for this game is a function $P : X \to \mathbb{R}$ such that for every player i and every $x_{-i} \in X_{-i}$,

$$P\left(x_i', x_{-i}\right) - P\left(x_i, x_{-i}\right) = u^i\left(x_i', x_{-i}\right) - u^i\left(x_i, x_{-i}\right) \text{ for all } x_i', x_i \in X_i.$$

A *potential game* is a game that has a potential function. If $x^* \in X$ maximizes the potential function for a game Γ then x^* is a pure strategy Nash equilibrium of Γ.

From this point forward, we suppose the strategy sets, $(X_i)_{i\in N}$, are convex and compact and that the payoff functions, $(u^i)_{i\in N}$, are continuously differentiable. Then P is a potential for Γ only if P is continuously differentiable, and

$$D_{x_i}u^i\left(x_i, x_{-i}\right) = D_{x_i}P\left(x_i, x_{-i}\right) \text{ for every } i \in N.$$

Given x_{-i}^*, a necessary condition for x_i^* to be an interior maximum of $u^i\left(x_i, x_{-i}^*\right)$ is $\nabla u^i\left(x_i^*, x_{-i}^*\right) = 0$. Noting that

$$\nabla P(x) = \begin{pmatrix} \nabla u^1(x) \\ \nabla u^2(x) \\ \vdots \\ \nabla u^n(x) \end{pmatrix},$$

it follows that x^* is an interior pure strategy Nash equilibrium only if $\nabla P(x^*) = 0$.

We now can apply Theorem 1 to provide new conditions under which a potential game has a unique equilibrium. Call Γ a *smooth potential game* if it has a smooth potential function.

Theorem 2 *Consider the smooth potential game Γ with potential P. Suppose the strategy sets, $(X_i)_{i\in N}$, are convex and compact, and that the payoff functions, $(u^i)_{i\in N}$, are continuously differentiable. Further suppose that ∇P is inward pointing on the boundary of X. Then Γ has a unique interior pure strategy Nash equilibrium if $\nabla P(x^*) = 0$ implies* $\det\left(-D^2P(x^*)\right) > 0$.

Proof. Any interior pure strategy Nash equilibrium is a critical point of P. By Theorem 1, P has a unique critical point on the interior of X which is also a maximum. Hence, this critical point is a Nash equilibrium. ■

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