Towson University
Department of Economics
Working Paper Series

Working Paper No. 2016-05

A Strong Correspondence Principle for Smooth, Monotone Environments

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March 2017

A Strong Correspondence Principle for Smooth, Monotone Environments

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March 10, 2017

Abstract

In discrete time dynamic systems that are locally monotone, we show that comparative statics are well-behaved if and only if equilibrium is exponentially stable. In addition, subject to boundary conditions but without local monotonicity, we show that the number of equilibria is finite and odd, and if every equilibrium is stable then there is exactly one. The results, which are applied to best response dynamics and adaptive dynamics, expand the scope of the correspondence principle to include a relationship between stability and uniqueness.

Keywords: correspondence principle, stability, comparative statics, uniqueness, best response dynamics, adaptive dynamics, $M-$matrix

JEL codes: D5, C6, C7

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1 Introduction

A model generates meaningful predictions if it excludes certain outcomes from occurring after one of the model's exogenous variables changes. Comparative statics, stability, and uniqueness play an essential role in determining whether a model is meaningful in this way.

In *Foundations* (1947), Samuelson proposed the stability hypothesis as a way to discipline comparative statics. Through a series of examples he argued that there exists an "intimate connection" between stability and comparative statics, a duality he termed the "correspondence principle."[1]

Two criticisms have been levied against Samuelson's version of the CP. First, Arrow and Hahn (1971, p. 321) noted that the CP had yet to live up to its intended purpose since the literature had not shown that stability was necessary for well-behaved comparative statics in any important class of models. Second, even if stability were necessary, this may not be sufficient to pin down comparative statics predictions when a model has multiple stable equilibria since the analysis is traditionally done locally via the Implicit Function Theorem (Kehoe, 1985). Thus, uniqueness is also a desirable trait for the refutability of an economic model.

Recently, several advances have been made in understanding the scope of the CP. In a seminal contribution, Echenique (2002) shows that well-behaved comparative statics and stability are equivalent in *globally* monotone models. Specifically, he proves the result under an impressively broad class of discrete time dynamics for globally increasing and self-mapping correspondences defined on rectangular regions of $\mathbb{R}^n$.[2,3] Kwong (2014) proves a version of this result for narrowly defined continuous time dynamics in smooth, cooperative systems.

We extend these results in two ways. First, we show that equivalence between stability in *discrete* time and comparative statics obtains for a broad class of smooth, *locally* monotone models which are not necessarily self-mapping and defined on pos-

[1] Samuelson had previously noted such a connection in earlier papers, but *Foundations* is where he coined the phrase the "correspondence principle."

[2] In fact, the result that ill-behaved comparative statics implies that the equilibrium is not stable holds in the space of lattices and under a very weak definition of stability. The converse requires the more restricted domain and a stronger definition of stability. Compare Theorems 2 and 4 in Echenique (2002).

[3] Echenique (2004) allows for non-continuous equilibrium selection functions, but requires more structure.

sibly non-rectangular regions of $\mathbb{R}^n$. Our approach is unique in the literature as we rely on the theory of $M-$matrices. Second, under weaker conditions we also show that there is a finite and odd number of equilibria, and that local stability implies uniqueness. The converse to the latter result exists in two dimensions if we impose local monotonicity. Alternatively, a partial converse due to Vives (1999, p. 54) exists in globally monotone models that map a rectangular region of $\mathbb{R}^n$ into itself. In these cases stability, uniqueness, and comparative statics are essentially equivalent.

Remarkably, the fact that stability implies uniqueness does not require the function to be self-mapping or monotone. We appeal to Poincaré-Hopf Theorem to establish this result, subject to regularity and boundary conditions (see Milnor, 1965).[4] One benefit of this approach is that we can simultaneously show that in general there is an odd and finite number of equilibria. Dierker (1972) and Hefti (2015) also take an index theory approach to derive related conclusions in specialized environments. Our treatment is perhaps more widely applicable since the scope of our analysis includes any model whose equilibrium is characterized by a system of equations. Also, while the Poincaré-Hopf Theorem applies to smooth manifolds, we take care to show that the corners of nonsmooth manifolds can be rounded. Thus, our result applies to domains which often arise in applications, like rectangles or simplexes.

2 An Illustration

After introducing some general notation we illustrate our results in two dimensions.

Consider the function $f : X \times T \to \mathbb{R}^n$ where $X \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^s$ are open sets. The function f is composed of n functions, $f_i : X \times T \to \mathbb{R}$ for $i = 1, ..., n$. We assume $s = 1$ but the results generalize to s finite. For a given $\bar{t} \in T$, $\bar{x}$ is an equilibrium if $f(\bar{x}; \bar{t}) = 0$. The equilibria of many economic models are characterized by such a system.

To be concrete, consider a parameterized two player game where each player $i = 1, 2$ selects a strategy $x_i \in \mathbb{R}_+$.[5] Given smooth and pseudoconcave payoff functions $F_i : \mathbb{R}^2_+ \times T \to \mathbb{R}$, define $f_i(x_i, x_j; t) \equiv \frac{\partial F_i}{\partial x_i}$. Then for a given $\bar{t} \in T$, interior Nash equilibria $\bar{x}$ are the solution to the system of first order conditions $f(\bar{x}; \bar{t}) = 0$.

[4]The key is to show that the index at a stable equilibrium is +1. Interestingly, McLennan (2016) suggests that there is an intimate connection between stability and an index of +1. Hence, the finding that stability implies uniqueness in this paper may be an example of a much deeper principle.

[5]We focus on interior equilibria so the domain of analysis is the open set $\mathbb{R}^2_{++}$.

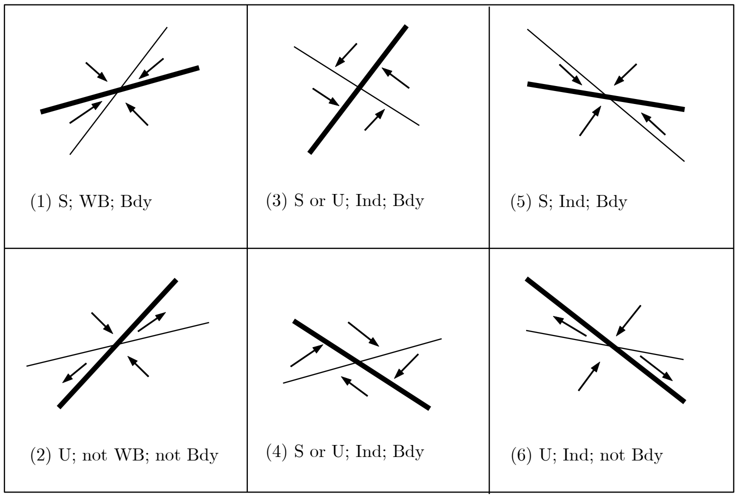

Figure 1: Types of Regular Equilibria.
Notes: The heavy line is player 2's best-response and the lighter line is player 1's, with x_2 on the vertical axis and x_1 on the horizontal axis. The abbreviations are: S - Stable; U - unstable; WB - well-behaved comparative statics; Ind - indeterminate comparative statics; Bdy - boundary condition satisfied.

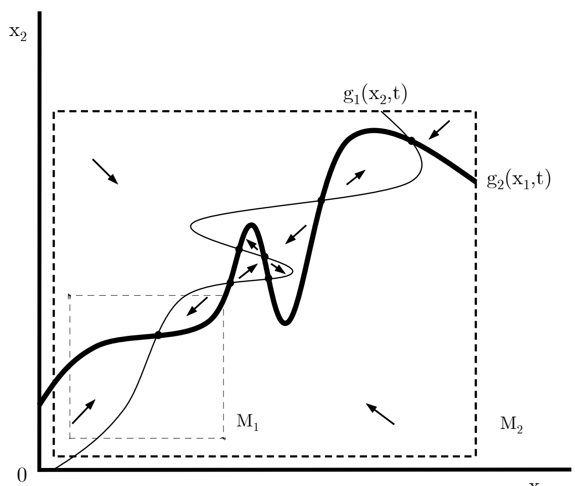

Figure 2: Best Response Functions.

Suppose we can write $f_i(x;t) = -x_i + g_i(x_j, t)$, $i \neq j$, so that $g_i : \mathbb{R}_+ \times T \to \mathbb{R}_+$ is player $i's$ best response function. Our concern is how the stability hypothesis relates to comparative statics and uniqueness. Letting τ be the time variable, we study (simultaneous) best response dynamics so that $x(\tau + 1) = g(x(\tau); t)$.

We assume all equilibria are regular. That is, at every equilibrium the Jacobian of f with respect to x, $D_x f(\bar{x}; \bar{t})$, is nonsingular. Non-regular equilibria arise if the best response functions are tangent to each other. Figure 1 illustrates the neighborhood around the six types of regular equilibria that can occur in this setting, assuming that x_1 is on the horizontal axis and x_2 is on the vertical axis. The stability properties of each are depicted with arrows and in the notes.

Suppose an increase in t increases the best response of each player. In Figures 1 and 2, this means g_1 (the lighter line) shifts to the right and g_2 shifts upwards. Comparative statics are *well-behaved* if the equilibrium unambiguously shifts to the northeast as a result. If the best response functions are locally monotone increasing as in panels (1)-(2) of Figure 1, comparative are well-behaved iff equilibrium is stable.

Analytically, the local stability of an equilibrium can be characterized through the Jacobian of $g = (g_1, g_2)$ with respect to x,

$$D_x g(x;t) = \begin{bmatrix} 0 & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & 0 \end{bmatrix}.$$

Recall that equilibrium is exponentially stable if the spectral radius[6] of $D_x g$, denoted $\rho(D_x g)$, is strictly less than one, unstable if $\rho(D_x g) > 1$, and may be stable or unstable if $\rho(D_x g) = 1$ (see Elaydi, 2005). By straightforward calculation, $\rho(D_x g) = \left| \left(\frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \right)^{1/2} \right|$, which is strictly less than one if and only if $\left| \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \right| < 1$.

Turning to comparative statics, at regular equilibria it follows from the Implicit Function Theorem (IFT) applied to $f(\bar{x}; \bar{t}) = 0$ that

$$\underbrace{\begin{bmatrix} \frac{d\bar{x}_1}{dt} \\ \frac{d\bar{x}_2}{dt} \end{bmatrix}}_{Dx(\bar{t})} = \underbrace{\begin{bmatrix} 1 & -\frac{\partial g_1}{\partial x_2} \\ -\frac{\partial g_2}{\partial x_1} & 1 \end{bmatrix}^{-1}}_{-D_x f(\bar{x};\bar{t})} \underbrace{\begin{bmatrix} \frac{\partial g_2}{\partial t} \\ \frac{\partial g_1}{\partial t} \end{bmatrix}}_{D_t f(\bar{x};\bar{t})}.$$

The *partial (or direct) effect* of a parameter change is $\frac{\partial g_i}{\partial t}$ for $i = 1, 2$. Thus, we say

[6] Letting $\lambda_1, ..., \lambda_n$ be the eigenvalues of a matrix A, the spectral radius of A is defined as $\rho(A) = \max\{|\lambda_1|, ..., |\lambda_n|\}$.

that comparative statics are well-behaved if $\frac{d\bar{x}_i}{dt} \geq 0$ for $i = 1, 2$ whenever $\frac{\partial g_i}{\partial t} \geq 0$ for $i = 1, 2$. Solving the system gives $\frac{d\bar{x}_i}{dt} = \frac{1}{1 - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1}}\left[\frac{\partial g_i}{\partial t} + \frac{\partial g_i}{\partial x_j}\frac{\partial g_j}{\partial t}\right]$. It follows that if best responses are nondecreasing, $\frac{\partial g_i}{\partial x_j} \geq 0$, $i \neq j$, then comparative statics are well-behaved iff $\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} < 1$.

In the knife-edge case, $\rho(D_x g) = 1$ iff $\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} = 1$, but then $D_x f(\bar{x}; \bar{t})$ is singular so that this is not a regular equilibrium. Hence, a regular equilibrium is exponentially stable iff comparative statics are well-behaved. Theorem 1 generalizes this result to $n-$dimensional discrete dynamic systems.

Regarding the number of equilibria, Theorem 2 says that if all equilibria within a contractible manifold M with boundary (e.g., a closed rectangle) are regular and a boundary condition is satisfied, then there exists an odd number of equilibria. In addition, if every equilibrium in M is stable then there is exactly one.

To understand the boundary condition, note that $f(x; \bar{t})$ defines a vector field over the strategy space that points in the same direction as best response dynamics. In this example the boundary condition requires that f points inward on the boundary of M (or equivalently that $-f$ points outward). In general, a vector field $f(x; \bar{t})$ *points inward* on the boundary of a manifold M if for any $\hat{x} \in bdy(M)$, $f(\hat{x}, \bar{t}) \neq 0$ and there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $\hat{x} + \varepsilon f(\hat{x}, \bar{t}) \in int(M)$. In this two player game, a sufficient condition for f to be inward pointing on the rectangle $M = [\underline{k}_1, \bar{k}_1] \times [\underline{k}_2, \bar{k}_2]$ is $g_i(x_j, \bar{t}) \in int(M)$ for all $x \in M$ and $i = 1, 2$. That is, each player $i's$ optimal choice is bounded below by $\underline{k}_i$ and above by $\bar{k}_i$ whenever $x \in M$. Equilibria cannot lie on the boundary.

Observe that any rectangle in Figure 2, such as M_1 or M_2, where f satisfies the boundary condition must contain an odd number of equilibria; and any such rectangle that contains a potentially stable equilibrium and at least one other equilibrium must contain an unstable equilibrium of the type (2) or (6).[7] Hence, if all equilibria are stable and the boundary condition is satisfied, then equilibrium is unique.

Analytically, subject to regularity and boundary conditions, the Poincaré-Hopf Theorem says that equilibrium is unique iff $\det D_x f(\bar{x}; \bar{t}) > 0$ at all equilibria. In this case, $\det D_x f(\bar{x}; \bar{t}) = 1 - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1}$, so it is easy to see that if all equilibria are stable then there is exactly one. In fact, for $n = 2$ if best responses are both nondecreasing or both nonincreasing then the converse is also true.

[7] In fact, we are not restricted to rectangles. These observations are valid for any contractible space with boundary.

3 The General Case

Let n be finite. While comparative statics of an equilibrium $\bar{x}$ are carried out via the IFT applied to $f(\bar{x};\bar{t}) = 0$, the intuition behind comparative statics results is typically sequential, and this implies some underlying dynamic process (Echenique, 2002). As in the previous illustration, the dynamic process is usually closely tied to f. To allow for broader application and more transparent proofs, however, we present our main results in the context of general dynamic systems. We provide additional applications in the next section where the dynamic system is generated from f.

Letting τ be the time variable and t the parameter, let dynamics be given by

$$x(\tau + 1) = g(x(\tau); t), \tag{1}$$

where $g : X \times T \to X$ is smooth. Define $h(x;t) \equiv x - g(x;t)$. Given $\bar{t} \in T$, note that $\bar{x}$ is an equilibrium of (1) if and only if $h(\bar{x};\bar{t}) = 0$. The local behavior of equilibrium is then determined by the IFT applied to h as long as equilibrium $\bar{x}$ is *$h-$regular*, that is, $\det D_x h\,(\bar{x};\bar{t}) \neq 0$. In the example from the previous section, $h = -f$.

In this framework, define the *partial effect* of an increase in t as $D_t g\,(\bar{x};\bar{t}) = \left(\frac{\partial g_1(\bar{x};\bar{t})}{\partial t}, ..., \frac{\partial g_n(\bar{x};\bar{t})}{\partial t}\right)^T$. This is the initial response of the dynamic system to the increase in t, after which the system evolves according to (1) with parameter $t + dt$. Say that *comparative statics are well-behaved* if the equilibrium does not decrease with t whenever the partial effect is nonnegative, that is, $Dx(\bar{t}) \geq 0$ whenever $D_t g\,(\bar{x}(\tau);t) \geq 0$.[8] Finally, we say that dynamics are *locally monotone* if the Jacobian of g at equilibrium is nonnegative, $D_x g(\bar{x};\bar{t}) \geq 0$. To see that this captures monotonicity, note that if $D_x g(\bar{x};\bar{t}) \geq 0$ for all $x \in X$ then g generates a monotone sequence in the sense that $\tilde{x}(\tau+1) \geq x(\tau+1)$ whenever $\tilde{x}(\tau) \geq x(\tau)$.

We need a few technical definitions and facts before stating our main results. The $n \times n$ matrix A is called an *M-matrix* if it can be written $A = \tau I - Y$ for some nonnegative matrix Y and scalar $\tau > \rho\,(Y)$, where $\rho(Y)$ is the spectral radius of Y. An $M-$matrix is a $Z-$matrix, where a *$Z-$matrix* is a square matrix with nonpositive off-diagonal elements. One of the many interesting characterizations of $M-$matrices is that if A is a $Z-$matrix, then it is an $M-$matrix if and only if

[8] Throughout the paper, for any matrix A, the the notation $A \geq 0$ means that each element of A is nonnegative. If y is a vector, then $y \geq 0$ means each element of y is nonnegative.

A^{-1} exists and $A^{-1} \geq 0$ (Plemmons, 1977). This last fact has been known in the economics literature as early as Debreu and Herstein (1953), but the application to the correspondence principle appears to be new.

As the illustration in the previous section showed, there is a knife-edge case which arises when the spectral radius of the Jacobian of the dynamic system equals one. The following result allows us to efficiently address this issue below.

Lemma 1 *Let A be an $n \times n$ real matrix and write $A = I - Y$ for $Y = I - A$. If $\rho(Y) \leq 1$, then $\det A \geq 0$, with strict inequality if $\rho(Y) < 1$. If, in addition, Y is nonnegative then $\rho(Y) = 1$ implies $\det A = 0$.*

Theorem 1 (Correspondence Principle) *Consider system (1). Suppose equilibrium $\bar{x}$ is $h-$regular and dynamics are locally monotone. Then comparative statics are well-behaved if and only if equilibrium is exponentially stable.*

Proof. To save notation, we drop the arguments of functions but the analysis is understood to take place at equilibrium.

If $\rho(D_x g) = 1$ then equilibrium is not regular. To see this, note that $D_x h = I - D_x g$ with $D_x g \geq 0$ by monotonicity, so $\det D_x h = 0$ by Lemma 1.

At regular equilibria, by the IFT

$$Dx(\bar{t}) = -[D_x h]^{-1} D_t h = -[I - D_x g]^{-1}[-D_t g] = [I - D_x g]^{-1} D_t g.$$

Then $Dx(\bar{t}) \geq 0$ whenever $D_t g \geq 0$ iff $[I - D_x g]^{-1} \geq 0$. By monotonicity, $I - D_x g$ is a $Z-$matrix and is an $M-$matrix iff $\rho(D_x g) < 1$. Thus, $[I - D_x g]^{-1} \geq 0$ iff $\rho(D_x g) < 1$.
■

The next result expands the scope of the correspondence principle by showing that if every equilibrium is exponentially stable then equilibrium is unique. This result does not require monotonicity, so in this sense the connection between stability and uniqueness seems to be more fundamental than the one between stability and comparative statics in Theorem 1. As a collateral result, we also provide conditions under which there exists an odd and finite number of equilibria.

Theorem 2 *Let $M \subset \mathbb{R}^n$ be a contractible manifold with differentiable boundary which is contained in a bounded open set $X \subset \mathbb{R}^n$. Fix $\bar{t}$ and let $h : X \times T \to \mathbb{R}^n$ be smooth. Let $\mathcal{E} = \{\bar{x} \in M | h(\bar{x}; \bar{t}) = 0\}$ be the set of equilibrium points in M. Suppose that all equilibria are regular and that h is outward pointing on the boundary of M.*

1. *The number of equilibria in $\mathcal{E}$ is finite and odd.*

2. *If every $\bar{x} \in \mathcal{E}$ is stable then $\{\bar{x}\} = \mathcal{E}$. That is, the equilibrium is unique.*

Proof. (1) By the IFT, the zeros of h are isolated since h is one-to-one in a neighborhood of each $\bar{x} \in M$. Note that $\mathcal{E} \subset X$ and X is certainly contained in a closed and bounded (and therefore compact) set. It then follows that $\mathcal{E}$ must be finite, since each point in $\mathcal{E}$ is isolated and compact sets satisfy the Bolzano-Weierstrass property – any infinite subset must accumulate.

Thus, the Poincaré-Hopf Theorem states that the index sum is equal to the Euler characteristic of M, which is $+1$ whenever M is a contractible subset of $\mathbb{R}^n$. When h is outward pointing on the boundary of M, the index of an equilibrium $\bar{x}$ is $+1$ if $\det D_x h > 0$ and -1 if $\det D_x h < 0$. Since $\det D_x h \neq 0$ and the index sum is $+1$, there must be an odd number of equilibria.

(2) Stability implies $\rho(D_x g) \leq 1$. Since $D_x h = I - D_x g$, it follows from Lemma 1 that $\det D_x h \geq 0$. Hence, the index at any regular, stable equilibrium is $+1$, and since the index sum is $+1$, equilibrium is unique. ■

Remark 1 *There are fixed point theorems in many settings. If a mapping $\varphi : X \to X$ is a* contraction mapping *(that is, there is a $0 \leq \alpha < 1$ such that for any x_1, x_2 in the domain, $d(\varphi(x_1), \varphi(x_2)) \leq \alpha d(x_1, x_2)$) then the Contraction Mapping Theorem (from the work of S. Banach circa 1920) implies that there is a unique equilibrium.*

In our setting, first-order local information *is given by the Jacobian $J_{\bar{x}}$ at each equilibrium $\bar{x}$, and we have assumed that $\rho(J_{\bar{x}}) \leq 1$. In the particular case that $\rho(J_{\bar{x}}) < 1$, then it is true that the mapping $x \mapsto J_{\bar{x}} x$ will be a contraction under* some metric *(possibly a rather different metric than the standard metric on $\mathbb{R}^n$). However, even if this is true at each equilibrium $\bar{x}$, there is no reason to believe that the global dynamics on X are given by a contraction.*

The beauty of the Poincaré-Hopf Theorem is that it allows us to come to these conclusions about global behavior using only local assumptions, and a reasonable boundary assumption.

3.1 Non-smooth Boundaries

A natural setting for applications is to have a space M which is either a simplex or a rectangular solid. In both these cases the boundary is not smooth, so at first glance it may seem that Theorem 2 does not apply. Fortunately this is not the case. In effect, the theorem applies equally well to the case where M is a simplex or is rectangular, as we will now explain.

There is a standard technique in differential topology that produces a *smoothing of* M – a manifold M' with differentiable boundary such that $M' \subset M \subset X$. We can construct M' in such a way that $\mathcal{E}' = \mathcal{E}$ (where $\mathcal{E}' = \{\bar{x} \in M' | h(\bar{x}; \bar{t}) = 0\}$), and so that h is outward (resp. inward) pointing on the boundary of M' if it is outward (resp. inward) pointing on the boundary of M. Thus we may use M' in place of M in Theorem 2 and our careful construction of M' implies that the conclusions of the theorem hold for M.

The method to define M' is the following. Choose a corner point $p \in M$ (by which we mean any non-smooth point on the boundary). In a neighborhood U of p, take a coordinate chart φ that identifies the boundary of M with the graph of some piecewise differentiable function $y : \mathbb{R}^{n-1} \to \mathbb{R}$, such that $\varphi(p) = (0, y(0)) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\varphi(x)$ is *below* this graph for points $x \in U \subset M$ which are not on the boundary.

Define $\mathfrak{y}(x)$ as the convolution of y with a bump function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ having the form

$$\psi(x) = \begin{cases} a \exp\left(-\frac{1}{1-|bx|^2}\right), & \text{if } |bx| < 1 \\ 0, & \text{otherwise} \end{cases}$$

for some $a, b > 0$. We state below how to choose b. Choose a such that the integral of $\psi(x)$ over $\mathbb{R}$ is 1. Notice the graph of $\mathfrak{y}$ agrees with the graph of y outside a neighborhood of 0. Let V be the intersection of $\varphi(U)$ with region in $\mathbb{R}^{n-1} \times \mathbb{R}$ having last coordinate on or below the graph of $\mathfrak{y}$. Then $\varphi^{-1}(V) \subset U$ will be the part of M' near p. Using a partition of unity, subordinate to a cover of the corner points, we can define the smoothing M'. Note that M' is contained in $int(M)$, the interior of M.

Having described how to construct M', we show that it is possible to make $\mathcal{E}' = \mathcal{E}$ and h inward pointing on the boundary of M' if it does so on the boundary of M (the outward pointing case is analogous). Let $d(q, r)$ be the standard Euclidean distance between two points $q, r \in \mathbb{R}^n$. Let $d(M, M')$ be defined as $\sup_{q \in M} \inf_{r \in M'} d(q, r)$.

In the construction of M', by choosing b sufficiently large (and adjusting a appropriately), we can make $d(M, M')$ arbitrarily small. Recall that if h points inward on the boundary of M then we can take a real-valued function $bdy(M) \to \mathbb{R}$ which at any $\hat{x} \in bdy(M)$, has value $\varepsilon_0 > 0$, and ε_0 has the property that for $0 < \varepsilon \leq \varepsilon_0$,

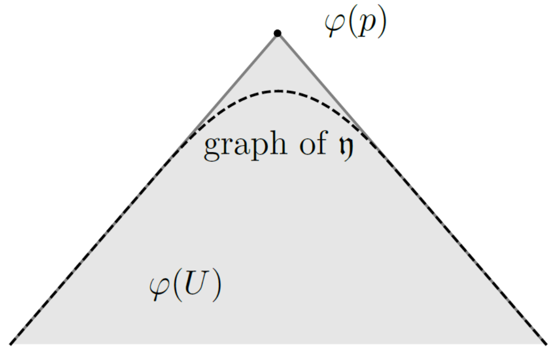

Figure 3: Rounding a corner p; U is a neighborhood of p

$\hat{x} + \varepsilon h(\hat{x}, \bar{t}) \in int(M)$. Continuity of h allows us to assume that the output ε_0 changes continuously as $\hat{x} \in bdy(M)$ varies. More generally, by considering vectors $x + \varepsilon h(x, \bar{t})$ for some $\varepsilon > 0$ and $x \in M - int(M')$, we will show that h points inward on the boundary of M', provided $d(M, M')$ is sufficiently small.

Given $\varepsilon > 0$ and $x \in X$ (and thinking of $\bar{t} \in T$ as fixed), use the short-hand notation $p(x, \varepsilon)$ for the vector $x + \varepsilon h(x, \bar{t})$. This defines a function $p : X \times \mathbb{R}_+ \to X$ and we note that p is continuous since h is continuous. By assumption, $p(\hat{x}, \varepsilon_0)$ is contained in the interior of M for any $\hat{x} \in bdy(M)$. Moreover, the function $p_0 : bdy(M) \to X$ which assigns $p_0(\hat{x}) = p(\hat{x}, \varepsilon_0)$ is continuous. The boundary of M is compact, so $d(p_0(\hat{x}),\ bdy(M))$ attains a minimum, say d_0. Choose b large enough so that $d(M, M') < d_0$. This guarantees that $p_0(\hat{x}) \in int(M')$ for each $\hat{x} \in bdy(M)$.

Since $int(M')$ is open, ε_0 varies continuously, and p is continuous, if $\hat{x} \in bdy(M)$ then $p(x, \varepsilon) \in int(M')$ for $x \in X$ as long as $d(x, \hat{x})$ and $|\varepsilon_0 - \varepsilon|$ are sufficiently small (ε_0 corresponding to the given $\hat{x}$). Thus, possibly by making $d(M, M')$ even smaller, we can guarantee that for each $x \in M - int(M')$, and for $x \in bdy(M')$ in particular, there is an $\epsilon > 0$ such that $p(x, \epsilon) \in int(M')$. As a consequence, h points inward on $bdy(M')$. Note this forces $h \neq 0$ on $M - int(M')$.

Since we have made $d(M, M')$ small enough that h is non-zero on the region $M - int(M')$, there are no equilibria in this region. Therefore, $\mathcal{E} \subset M'$ and $\mathcal{E}' = \mathcal{E}$.

4 Applications

Two important classes of dynamics are adaptive dynamics and best response dynamics. In this section we explore how our results apply in these cases.

Adaptive Dynamics. Adaptive dynamics require that x_i increases only if the function f_i is positive. Formally, *adaptive dynamics* are modeled by the following set of difference equations:

$$x_i(\tau+1) = x_i(\tau) + k_i(f_i(x(\tau);t)) \quad \text{for } i = 1, ..., n, \tag{2}$$

where $k_i : \mathbb{R} \to \mathbb{R}_{++}$ is a strictly increasing, smooth function. In matrix-vector notation this system is written $x(\tau+1) = x(\tau) + k(f(x(\tau);t))$.

If f_i is the marginal net benefit of increasing an action, then adaptive dynamics say that the action increases only if the marginal net benefit is positive. Alternatively, in a general equilibrium setting adaptive dynamics may be interpreted as a discrete time version of tâtonnement dynamics. When k is the identity function this is also a discrete time version of the continuous time dynamics considered in Kwong (2014).

In the context of the general case considered in Section 3, $g(x(\tau);t) = x(\tau) + k(f(x(\tau);t))$ and $h(x,t) = -k(f(x;t))$. Likewise, the partial effect and Jacobian of g are, respectively,

$$D_t g = DkD_t f \quad \text{and} \quad D_x g = I + DkD_x f,$$

where Dk is the diagonal matrix whose typical (i,i) entry is $\frac{\partial k_i(f_i(\bar{x};\bar{t}))}{\partial f_i} > 0$. Since $D_x h = -DkD_x f$ and $Dk \neq 0$, it follows that equilibrium is $h-$regular iff it is $f-$regular in the sense that $\det D_x f(\bar{x};\bar{t}) \neq 0$. Moreover, the local behavior of equilibrium can be determined by applying the IFT to f or h since $D_t h = -DkD_t f$ and

$$Dx(\bar{t}) = -\left[D_x f\right]^{-1} D_t f = -\left[DkD_x f\right]^{-1} DkD_t f = -\left[D_x h\right]^{-1} D_t h.$$

Then Theorem 1 applies if we replace the condition that equilibrium is $h-$regular with $f-$regular. Theorem 2 applies if we replace h everywhere with f and assume f is inward pointing on the boundary (or equivalently that $-f$ is outward pointing). The first part of Theorem 2 follows immediately, but for the second part we need to show that $\rho(D_x g) \leq 1$ implies $\det(-D_x f) \geq 0$. We know from the proof that $\rho(D_x g) \leq 1$ implies $\det D_x h = \det(-DkD_x f) \geq 0$. Since $\det Dk > 0$ it follows that

$\det(-D_x f) \geq 0$. Note that the illustration from Section 2 is a special case of adaptive dynamics where k is the identity function.

To interpret the conditions of Theorem 1 in this context, note that the partial effect is nonnegative if $D_t f(\bar{x};\bar{t}) \geq 0$ while local dynamics are monotone if, for $i = 1, ..., n$, $\frac{\partial f_i}{\partial x_i} \geq -1/\frac{\partial k_i}{\partial f_i}$ and $\frac{\partial f_i}{\partial x_j} \geq 0$ for $i \neq j$. These conditions have a natural interpretation when $f(\bar{x};\bar{t}) = 0$ is the system of first order conditions for an interior solution to an optimization problem. If $F(x;t)$ is the objective function then the partial effect is nonnegative if $F(\bar{x};\bar{t})$ has increasing differences in $(x;t)$ and dynamics are monotone if $F(\bar{x};\bar{t})$ is supermodular in x and $\frac{\partial f_i}{\partial x_i} \geq -1/\frac{\partial k_i}{\partial f_i}$ for all i. This interpretation extends to a strategic setting where F is the payoff function of any given player.

Best Response Dynamics. Given our equilibrium system $f(x;t) = 0$, *best response dynamics* require that for any $x(\tau)$, each $x_i(\tau+1)$ is chosen to set f_i to zero given $x_{-i}(\tau) = (x_1(\tau), ..., x_{i-1}(\tau), x_{i+1}(\tau), ..., x_n(\tau))$. The term "best response dynamics" is motivated by a strategic setting where f is a system of best response functions, one for each player, as in the example of the Section 2. In a general equilibrium setting where the vector x represents prices and f is the excess demand function, this describes a scenario in which, for each market, the price is set to clear the market assuming that the prices in other markets remain at their levels from the previous period.

To deal with the possibility that $f(x;t)$ may be implicitly defined, transform $f(x;t)$ locally into an explicitly defined system of equations via the IFT as follows. Fix $\bar{t} \in T$ and define $z : X \times X \times T \to \mathbb{R}^n$ as

$$z(x(\tau+1); x(\tau), \bar{t}) = 0 \tag{3}$$

where, for each $i = 1, ..., n$, $z_i(x(\tau+1); x(\tau), \bar{t}) = f_i(x_i(\tau+1); x_{-i}(\tau), \bar{t})$.

Note that $\bar{x}$ is an equilibrium of system (3) if and only if it is an equilibrium of the system $f(x;t) = 0$. Intuitively, in each component equation z_i, the values of $x_{-i}(\tau)$ are taken as given and are treated as parameters. Noting that X is open and assuming $\frac{\partial f_i(\bar{x};\bar{t})}{\partial x_i} \neq 0$ for all i, the IFT implies there exists a unique function g defined in an open neighborhood around $\bar{x}$ such that

$$x(\tau+1) = g(x(\tau), \bar{t}).$$

This is the equivalent of system (1) in the general case, although here it is locally defined, as is $h(x,t) = x - g(x;t)$. This is sufficient since stability and comparative statics are local properties.

Let Λ be the diagonal matrix with diagonal entries $(1/\frac{\partial f_1}{\partial x_1}, 1/\frac{\partial f_2}{\partial x_2}, ..., 1/\frac{\partial f_n}{\partial x_n})$. By another application of the IFT, the partial effect and Jacobian of g are, respectively,[9]

$$D_t g = -\Lambda D_t f \text{ and } D_x g = I - \Lambda D_x f(\bar{x};\bar{t}).$$

Since $D_x h = I - D_x g = \Lambda D_x f(\bar{x};\bar{t})$ and $\det \Lambda \neq 0$, equilibrium is $h-$regular iff it is f-regular in the sense that $\det D_x f(\bar{x};\bar{t}) \neq 0$. Moreover, the local behavior of equilibrium can be determined by applying the IFT to f or h since

$$Dx(\bar{t}) = -\left[D_x f\right]^{-1} D_t f = -\left[\Lambda D_x f\right]^{-1} \Lambda D_t f = -\left[D_x h\right]^{-1} D_t h.$$

Then Theorem 1 applies if we replace the condition that equilibrium is $h-$regular with $f-$regular. Theorem 2 applies if we replace h everywhere with f; one may assume f is inward pointing or outward pointing as is convenient. In either case, the first part of the theorem follows immediately. If f is outward pointing, then the second part of the theorem also requires the condition that $\Pi_{i=1}^n \frac{\partial f_i}{\partial x_i} > 0$ at every equilibrium. This condition is required to ensure that stability implies $\det D_x f \geq 0$; we know from the proof that stability implies $\det D_x h = \det \Lambda D_x f \geq 0$, and if $\det \Lambda > 0$ it follows that $\det D_x f \geq 0$. If f is inward pointing (so that $-f$ is outward pointing), we need $\Pi_{i=1}^n - \frac{\partial f_i}{\partial x_i} > 0$ at every equilibrium since $\det D_x h = \det(-\Lambda)\det(-D_x f)$. In the illustration from Section 2, f is assumed inward pointing and the diagonal entries of Λ are $(-1,-1)$, so stability implies $\det(\Lambda D_x f) = \det(-D_x f) \geq 0$, as desired.

To interpret the conditions of Theorem 1, observe that nonnegative partial effects means $-\frac{\partial f_i}{\partial t}\frac{1}{\partial f_i/\partial x_i} \geq 0$ for $i = 1, ..., n$ and dynamics are locally monotone if $-\frac{\partial f_i/\partial x_j}{\partial f_i/\partial x_i} \geq 0$, $i \neq j$. To see that these conditions are natural, totally differentiate the system $f(\bar{x};\bar{t}) = 0$ to get

$$\frac{dx_i(\bar{t})}{dt} = -\frac{\partial f_i}{\partial t}\frac{1}{\partial f_i/\partial x_i} + \sum -\frac{\partial f_i/\partial x_j}{\partial f_i/\partial x_i}\frac{dx_j(\bar{t})}{dt}, \; i = 1, ..., n. \tag{4}$$

The first term on the right hand side of this equality is the partial effect on $\bar{x}_i$ while

[9] By the IFT, the partial effect is $D_t g = -[D_{x(\tau+1)}z]^{-1} D_t z$. By definition, $-[D_{x(\tau+1)}z]^{-1} = -[\Lambda^{-1}]^{-1} = -\Lambda$ and $D_t z = D_t f$. Similarly, the Jacobian is $D_x g = -[D_{x(\tau+1)}z]^{-1} D_{x(\tau)}z$, but given the definitions of z_i and Λ, it follows that $[D_{x(\tau+1)}z]^{-1} = \Lambda$ and $D_{x(\tau)}z = D_x f - \Lambda^{-1}$, so $D_x g = -\Lambda(D_x f - \Lambda^{-1}) = -\Lambda D_x f + I$.

$-\frac{\partial f_i/\partial x_j}{\partial f_i/\partial x_i}$ describes how the best response changes due to a unit increase in x_j. In matrix notation we can write (4) as $Dx(\bar{t}) = D_t g + D_x g Dx(\bar{t})$, which additively decomposes the effect of a parameter change into its partial effect and the subsequent interactions effect as the system adjusts to its new equilibrium. Thus, in a strategic setting, nonnegative partial effects means that an increase in t does not decrease any players' best response, and local dynamics are monotone if the players' best response functions are nondecreasing.

5 Conclusion

This note has proven a correspondence principle using local properties of a discrete time dynamic system. We also use local properties to show that stability implies uniqueness even without the monotonicity assumption. The results were shown to apply to best response dynamics and adaptive dynamics. These findings suggest that perhaps the correspondence principle should be understood to refer to an "intimate connection" between stability, comparative statics, *and* uniqueness.

6 Appendix

Proof of Lemma 1.. Let $\sigma(Y) = \lambda_1, ..., \lambda_n$ denote the spectrum of Y and $\sigma(A)$ the spectrum of A. Since $A = I - Y$ it follows that $1 - \lambda_i \in \sigma(A)$ iff $\lambda_i \in \sigma(Y)$.

If λ_i is real-valued, then $|\lambda_i| \leq (<)1$ implies $1 - \lambda_i \geq (>)0$. If λ_j is complex-valued, then since the eigenvalues are the roots of a characteristic polynomial with real coefficients, the conjugate pair of λ_j is also an eigenvalue. Consequently, $1 - \lambda_j$ and its conjugate pair belong to $\sigma(A)$. Since the product of a complex number and its conjugate pair is positive, it follows that $\det A = \Pi_{i=1}^n (1 - \lambda_i) \geq 0$, with strict inequality iff $\lambda_i \neq 1$ for $i = 1, ..., n$.

If $Y \geq 0$, then Theorem 1.7.3 in Bapat and Raghavan (1997), which extends some of the conclusions of the Perron-Frobenius theorem to reducible matrices, implies that 1 is an eigenvalue of Y if $\rho(Y) = 1$. Hence, A possesses a zero eigenvalue. Since $\det D \neq 0$, $\det DA = \det D \det A$, and the determinant of a matrix equals the product of its eigenvalues, this implies A possesses a zero eigenvalue. Hence, $\det A = 0$. ■

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